On the factorization invariants of arithmetical congruence monoids

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Theorem

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What about algebraic structures exhibiting non-unique factorization?

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Example

The atoms of $\mathbb{N}$ are the prime numbers $\mathbb{P}$.

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A **factorization** of an element $x \in M$ is a product $x = a_1 a_2 \cdots a_n$ where $a_1, a_2, \ldots, a_n \in \mathcal{A}(M)$.

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For $x \in M$, we denote $\mathsf{Z}(x)$ to be the set of factorizations of x.

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  - Similarly, the atoms of $2\mathbb{N}_0 + 1$ are $\mathbb{P}\backslash\{2\}$. So, $2\mathbb{N}_0 + 1$ is also a UFM by the Fundamental Theorem of Arithmetic.

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Note that for two elements $1 + 4k_1, 1 + 4k_2 \in M$, we have

$$(1 + 4k_1)(1 + 4k_2) = 1 + 4k_1 + 4k_2 + 16k_1k_2 = 1 + 4(k_1 + k_2 + 4k_1k_2)$$

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The element 1 serves as the identity element.

This shows that $\{1 + 4k \mid k \in \mathbb{N}_0\}$ under multiplication is a monoid.

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Note that the element $693 = 1 + 4 \cdot 173 \in M = \{1 + 4k \mid k \in \mathbb{N}_0\}$ can be factored as

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where $9, 21, 33, 77 \in \mathcal{A}(M)$.

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Thus, Hilbert's monoid is not a UFM.

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$$\{a, a+b, a+2b, a+3b, \ldots\} \cup \{1\} = (a + b\mathbb{N}_0) \cup \{1\}$$

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- $\mathbb{N} = M_{1,1}$
- Hilbert's monoid: $\{1 + 4k \mid k \in \mathbb{N}_0\} = M_{1,4}$
- Meyerson's monoid: $\{1\} \cup \{4 + 6k \mid k \in \mathbb{N}_0\} = M_{4,6}$

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Remark

Note that all regular ACMs must be multiplicatively closed, implying that $b \mid a^2 - a = a(a-1)$. But $\gcd(a, b) = 1$ and $a \leq b$, and thus $a = 1$. So, all regular ACMs will take the form $M_{1,b}$.

Monoid invariants measure how far a monoid is from being a UFM.
In our talk, we will define and compute length density, which is one of three factorization invariants of ACMs we studied.

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In $\mathbb{N}$, the length set of any $x \in \mathbb{N}$ contains 1 element.

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Thus, $\mathsf{L}(10000) = \{3, 4\}$.

Definition

Consider $x \in M$ for a monoid M. Let $\mathrm{L}(x) = \{n_1, n_2, \ldots n_k\}$ where $n_1 < n_2 < \cdots < n_k$. Then, the **delta set of** x is defined to be

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Example

If an element $x \in M$ has $\mathsf{L}(x) = \{2, 5, 7, 11\}$, we have

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The length density measures how sparse the distribution of the factorization lengths are.

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In contrast, if an element $x \in M$ has $\mathsf{L}(x) = \{2, 3, 4, 5, 7, 8, 9, 11\}$, we have

$$\mathsf{LD}(x) = \frac{|\mathsf{L}(x)| - 1}{\max \mathsf{L}(x) - \min \mathsf{L}(x)} = \frac{8-1}{11-2} = \frac{7}{9}.$$

The following result reveals an interaction between the length density and delta sets.

Theorem (Chapman, O'Neill, and Ponomarenko, 2022)

For a monoid M and element $x \in M^{LI}$, we have

$$\frac{1}{\max \Delta(x)} \leq \mathrm{LD}(x) \leq \frac{1}{\min \Delta(x)}.$$

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Theorem (Liu, Ma, and Zhang, 2022)

Let $M_{1,b}$ be a regular ACM. Then

$$\mathrm{LD}(M_{1,b}) = \begin{cases} \varnothing & \phi(b) \leq 2 \\ \frac{1}{\phi(b)-2} & \phi(b) \geq 3 \end{cases}.$$

Example

For the monoid $M_{1,7}$, which is the set $\{1 + 7k \mid k \in \mathbb{N}_0\}$, we have that the element 15^6 can only be factored into $3^6 \cdot 5^6$ and $(15)^6$.

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This means $\mathsf{L}(15^6) = \{2, 6\}$, implying $\mathsf{LD}(15^6) = \frac{1}{4}$ and $\mathsf{LD}(M_{1,7}) \leq \frac{1}{4}$.

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This means $\mathsf{L}(15^6) = \{2, 6\}$, implying $\mathsf{LD}(15^6) = \frac{1}{4}$ and $\mathsf{LD}(M_{1,7}) \leq \frac{1}{4}$.

We can also prove by contradiction that $\frac{1}{4} \leq \frac{1}{\max \Delta(x)}$. So, $\mathsf{LD}(M_{1,7}) \geq \frac{1}{4}$. This forces $\mathsf{LD}(M_{1,7}) = \frac{1}{4}$.

We now discuss $\mathrm{LD}(M_{a,b})$ where $\gcd(a,b) = p^\alpha$ for $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$. Let β denote the least integer such that $p^\beta \in M$. Let $\delta(\alpha,\beta)$ denote the largest integer less than $\frac{\beta}{\alpha}$.

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Example

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Theorem (Liu, Ma, and Zhang, 2022)

For a local ACM $M_{a,b}$, the length density can be characterized as

$$\mathrm{LD}(M_{a,b}) = \begin{cases} \varnothing & \text{if } \alpha = \beta = 1 \\ 1 & \text{if } \alpha = \beta > 1 \\ \frac{1}{\delta(\alpha,\beta)} & \text{if } \alpha < \beta \end{cases}.$$

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Since $\gcd(4, 6) = 2^1$, $\alpha = 1$. Also, $\beta = 2$ since $4 \in M_{4,6}$.

It has previously been shown that when $\alpha < \beta$, we have $\Delta(M_{4,6}) = [1, \frac{\beta}{\alpha})$. Thus, $\Delta(M_{4,6}) = [1, 2) = \{1\}$.

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We also have that $\frac{1}{\max \Delta(x)} \leq \mathrm{LD}(x)$. Thus, $1 \leq \mathrm{LD}(M_{4,6})$.

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Thus, $\mathsf{L}(10000) = \{3, 4\}$ which implies $\mathsf{LD}(10000) = 1$.

By the definition of length density, $\mathsf{LD}(M_{4,6}) \leq 1$. So, our two bounds force $\mathsf{LD}(M_{4,6}) = 1$.

References I

D. Anderson, D. F. Anderson, and M. Zafrullah.
Factorization in integral domains.
J. Pure Appl. Algebra, 69(1):1–19, 1990.

D. F. Anderson, S. T. Chapman, N. Kaplan, and D. Torkornoo.
An algorithm to compute ω-primality in a numerical monoid.
In *Semigroup Forum*, volume 82, pages 96–108. Springer, 2011.

P. Baginski and S. T. Chapman.
Arithmetic congruence monoids: a survey.
In *Combinatorial and Additive Number Theory*, pages 15–38. Springer, 2014.

P. Baginski, S. T. Chapman, and G. J. Schaeffer.
On the delta set of a singular arithmetical congruence monoid.
J. Theor. Nr. Bordx., 20(1):45–59, 2008.

M. Banister, J. Chaika, S. T. Chapman, and W. Meyerson.
On the arithmetic of arithmetical congruence monoids.
In *Colloq. Math.*, volume 1, pages 105–118, 2007.

M. Banister, J. Chaika, S. T. Chapman, and W. Meyerson.
A theorem on accepted elasticity in certain local arithmetical congruence monoids.
Abh. Math. Semin. Univ. Hambg, 79(1):79–86, 2009.

T. Barron, C. O'Neill, and R. Pelayo.
On dynamic algorithms for factorization invariants in numerical monoids.
Math. Comput., 86(307):2429–2447, 2017.

C. Brower, S. T. Chapman, T. Kulhanek, J. McDonough, C. O'Neill, V. Pavlyuk, and V. Ponomarenko.
Length density and numerical semigroups.
Preprint. Available at arXiv:2110.10618, 2021.

S. T. Chapman.
A tale of two monoids: A friendly introduction to nonunique factorizations.
Math. Mag., 87(3):163–173, 2014.

S. T. Chapman, P. A. García-Sánchez, Z. Tripp, and C. Viola.
Measuring primality in numerical semigroups with embedding dimension three.
J. Algebra Appl., 15(01):1650007, 2016.

S. T. Chapman, C. O'Neill, and V. Ponomarenko.
On length densities.
In *Forum Math.*, volume 34, pages 293–306. De Gruyter, 2022.

S. T. Chapman and D. Steinberg.
On the elasticity of generalized arithmetical congruence monoids.
Results Math., 58(3):221–231, 2010.

S. T. Chapman and Z. Tripp.
ω-primality in arithmetic leamer monoids.
In *Semigroup Forum*, volume 99, pages 47–56. Springer, 2019.

H. Davenport.
The higher arithmetic: An introduction to the theory of numbers.
Cambridge University Press, 1999.

A. Fujiwara, J. Gibson, M. O. Jenssen, D. Montealegre, V. Ponomarenko, and A. Tenzer.
Arithmetic of congruence monoids.
Commun. Algebra, 44(8):3407–3421, 2016.

A. Geroldinger.
Chains of factorizations in weakly krull domains.
In *Colloq. Math.*, volume 72, pages 53–81, 1997.

A. Geroldinger, A. J. Grynkiewicz, and W. A. Schmid.
The catenary degree of krull monoids i.
J. Theor. Nr. Bordx., 23(1):137–169, 2011.

A. Geroldinger and F. Halter-Koch.
Non-unique factorizations: Algebraic, combinatorial and analytic theory.
Chapman and Hall/CRC, 2006.

A. Geroldinger and W. Hassler.
Local tameness of v-noetherian monoids.
J. Pure Appl. Algebra, 212(6):1509–1524, 2008.

A. Geroldinger and Q. Zhong.
The catenary degree of krull monoids ii.
J. Aust. Math. Soc., 98(3):324–354, 2015.

F. Halter-Koch.
Arithmetical semigroups defined by congruences.
In *Semigroup Forum*, volume 42, pages 59–62. Springer, 1991.

References VIII

J. Hartzer and C. O'Neill.
On the periodicity of irreducible elements in arithmetical congruence monoids.
Preprint. Available at arXiv:1606.00376, 2016.

W. Narkiewicz.
On algebraic number fields with non-unique factorization, ii.
In *Colloq. Math.*, volume 15, pages 49–58. Institute of Mathematics Polish Academy of Sciences, 1966.

M. Omidali.
The catenary and tame degree of numerical monoids generated by generalized arithmetic sequences.
2012.

C. O'Neill.
On factorization invariants and hilbert functions.
J. Pure Appl. Algebra, 221(12):3069–3088, 2017.

C. O'Neill and R. Pelayo.
How do you measure primality?
Amer. Math. Monthly, 122(2):121–137, 2015.

C. O'Neill and R. Pelayo.
Realisable sets of catenary degrees of numerical monoids.
Bull. Aust. Math. Soc., 97(2):240–245, 2018.

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