# Computing Truncated Metric Dimension on Trees (mentor Zi Song Yeoh)

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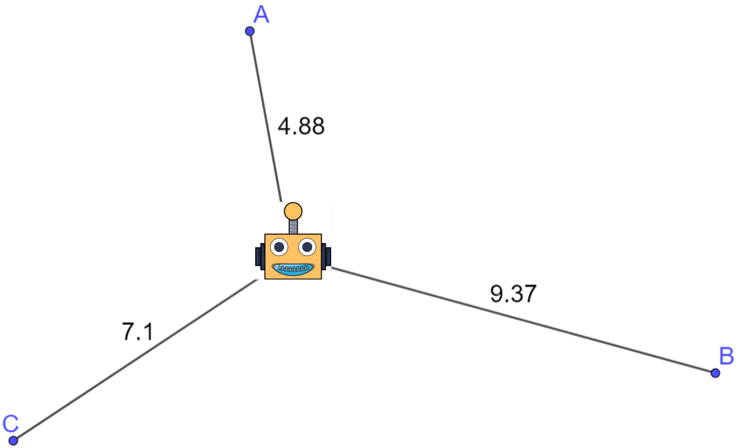
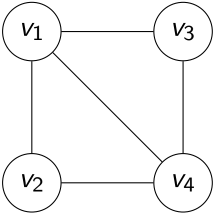
# Introduction

In the Euclidean plane, for any set of three non-collinear points, any point in the plane can be determined solely by its distances to the points in the set.



# Graph Distance

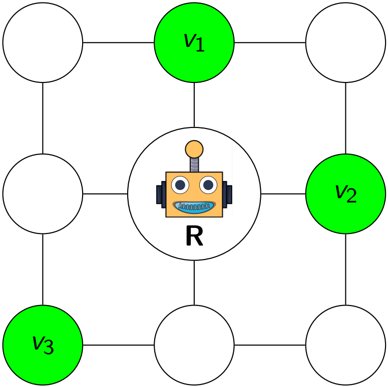
We let $G = (V, E)$ be a finite, simple, connected graph. All edges have length 1. Let $d(u, v)$ denote the shortest distance between vertices $u, v$.




$$d(v_1, v_4) = 1, \ d(v_2, v_3) = 2$$

# Introduction (continued)

Now, let the robot move from vertex to vertex on a graph $G$. Let $R$ denote the vertex the robot is on.




$$d(R, v_1) = 1,\ d(R, v_2) = 1,\ d(R, v_3) = 2$$

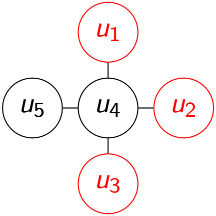
# Resolving Set

**Definition (Resolving Set)**

Let $S = \{z_1, z_2, \ldots, z_m\}$ be a subset of $V(G)$. For every vertex $x \in V(G)$, create a tuple

$$\alpha(x) = (d(x, z_1), d(x, z_2), \ldots, d(x, z_m)).$$

If $\alpha(x)$ is distinct for every $x \in V(G)$, then we say $S$ is a resolving set of $G$.




$$S = \{u_1, u_2, u_3\}$$

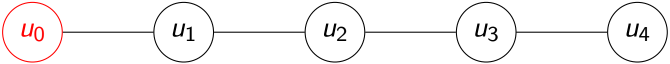
$$\alpha(u_1) = (d(u_1, u_1), d(u_1, u_2), d(u_1, u_3)) =$$

$$= (0, 2, 2)$$

$$\alpha(u_2) = (2, 0, 2),\ \alpha(u_3) = (2, 2, 0)$$

$$\alpha(u_4) = (1, 1, 1),\ \alpha(u_5) = (2, 2, 2)$$

# Resolving Set Example (Path)




$$S = \{u_0\}$$

$$\alpha(u_0) = (0)$$

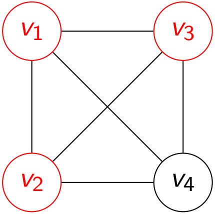
$$\alpha(u_1) = (1)$$

$$\alpha(u_2) = (2)$$

$$\alpha(u_3) = (3)$$

$$\alpha(u_4) = (4)$$

# Resolving Set Example (Clique)




$$S = \{v_1, v_2, v_3\}$$

$$\alpha(v_1) = (0, 1, 1),\ \alpha(v_2) = (1, 0, 1)$$

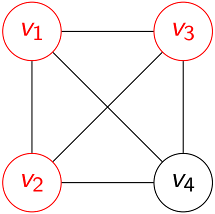
$$\alpha(v_3) = (1, 1, 0),\ \alpha(v_4) = (1, 1, 1)$$

# Metric Dimension

**Definition (Metric Dimension)**

The size of the smallest resolving set of $G$ is called the metric dimension of $G$, and it is denoted $\dim(G)$.

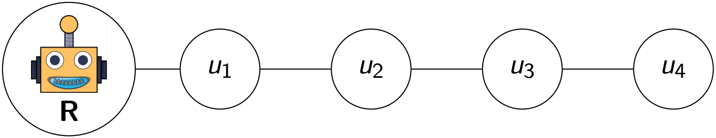
**Example**




$$\dim(G) = 3$$

# $k$-Truncated Distance

What if the robot's sensors have a finite range?

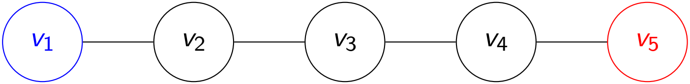



Let $k$ be an arbitrary positive integer. Define the $k$-truncated distance, $d_k(u, v) := \min(d(u, v), k + 1)$. In our example, let $k = 2$, and let $R$ be the vertex the robot is on. Then we have:

$$d_k(R, u_1) = 1,\ d_k(R, u_2) = 2,\ d_k(R, u_3) = 3,\ d_k(R, u_4) = 3$$

# $k$-Truncated Resolving Set

When we consider $k$-truncated distance instead of regular distance, the resolving set is called the $k$-truncated resolving set, and the metric dimension is called the $k$-truncated metric dimension, denoted $\dim_k(G)$.




For the above example, let $k = 2$. Let $S = \{v_1\}$, $S' = \{v_1, v_5\}$. Both $S$ and $S'$ are resolving sets of the graph. However, $S$ is not a $k$-truncated resolving set of the graph because

$$d_k(v_1, v_4) = d_k(v_1, v_5) = 3.$$

Note that $S'$ is a $k$-truncated resolving set of the graph.

# Past Results

- It is known that computing $\dim(G)$ and $\dim_k(G)$ for general graphs $G$ is NP-Hard (Estrada-Moreno, Yero, Rodrigueq-Velazquez)
- For trees $T$, computing $\dim(T)$ can be done in linear time (Khuller, Raghavachari, Rosenfeld)
- For trees $T$, computing $\dim_1(T)$ can be done in linear time, using dynamic programming (Frongillo, Lladser, Tillquist)

# Our Results

In our paper, we focused on algorithms for computing $k$-truncated metric dimension in trees. We proved the following two results:

- Computing $\dim_k(T)$ for general $k, n$ is NP-Hard
- If $k$ is fixed, then there exists an algorithm to compute $\dim_k(T)$ with time complexity polynomial in $n$

# NP-Hardness

The 3-dimensional matching (3DM) problem is known to be NP-Hard. Our approach to show that computing $\dim_k(T)$ for general $n, k$ is NP-Hard was via a technique called reduction. We showed that:

$$\dim_k(T) \text{ can be computed in polynomial time } \implies$$

$$\implies \text{ 3DM can be solved in polynomial time}$$

# Conclusion

Many open questions remain regarding the computation of $k$-truncated metric dimension. For example,

- What is the best dependence on $k$ we can get in an algorithm to compute $\dim_k(T)$?
- What is the best approximation ratio we can obtain for $k$-truncated metric dimension of trees?
- Can we efficiently compute truncated metric dimension in other classes of graphs for any constant $k$?

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