MIT PRIMES-USA Conference

10/15/2022

Unitary Conditions of Heun and Lamé Differential Equations

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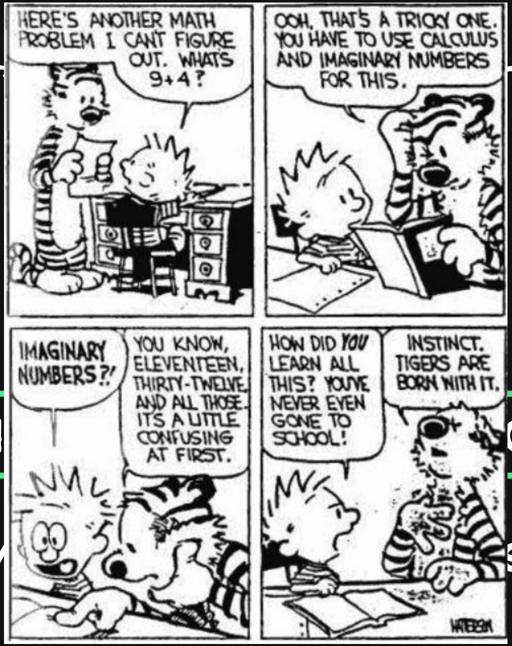
Table of contents

01

Complex Analysis Crash Course



03

Our Project

04

Analy

05

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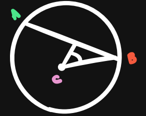
01

Complex Analysis Crash Course

Don’t worry, there are emergency supplies if you do crash.

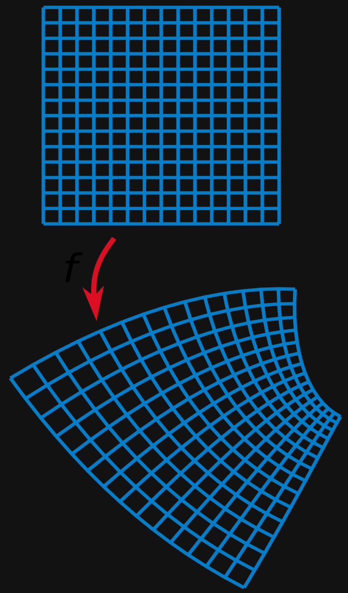


What is Complex Analysis?

- TLDR: Calculus with complex numbers
- Intuitively similar to multivariable calculus (imagine two dimensions, one for the real component and one for the imaginary)
- Many real functions $(e^x,\ \sin(x), \text{etc.})$ can be nicely extended to the complex plane.

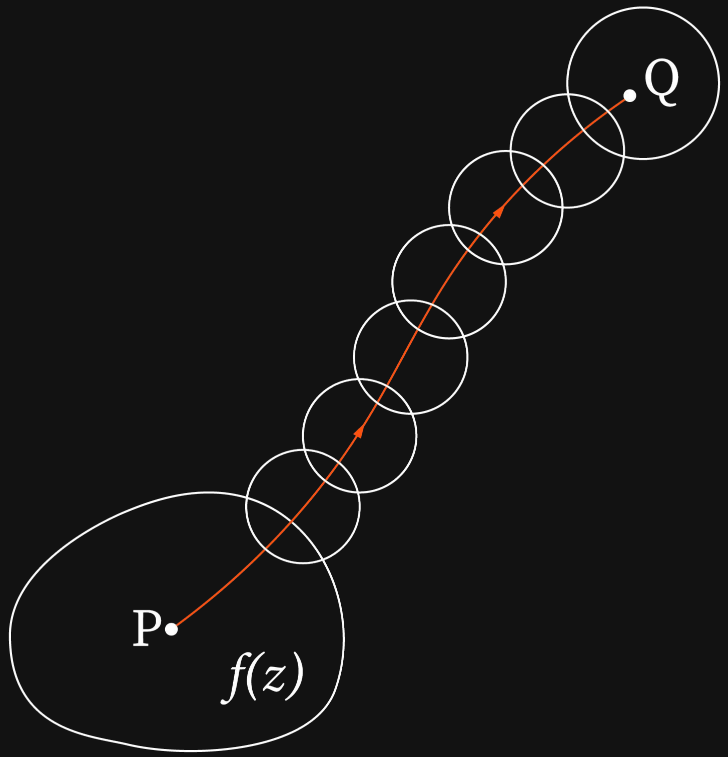
Complex Derivative

- We define a complex derivative as $f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$
- Unlike multivariable partial derivatives, with the complex derivative, it does not depend on the direction from which h approaches 0.
- Therefore, complex differentiable functions locally look like just a rotation and scaling of the complex plane.
- Complex differentiable functions are called **analytic**.



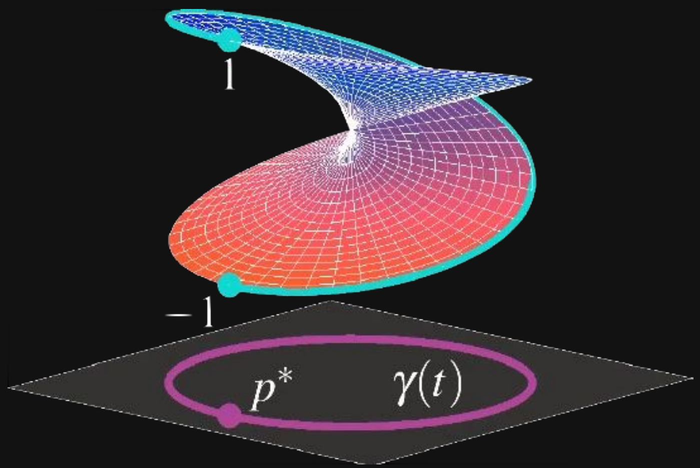
Principle of Analytic Continuation

- Start with an **analytic** (i.e differentiable, expressible as a power series) function f at a point
- There is a little area around P where f is analytic
- We can extend this along a curve to point Q to get a value for $f(Q)$
- Doesn't matter which path is taken (**usually**)



What about Loops?

- This value is not uniquely-defined if f is not analytic inside the loop.
- If f is not analytic inside a loop, we get a matrix mapping $f \to Mf$ by going around the loop once.
- By going around the opposite direction, we invert this map.



$C_y = C_y^{\alpha}(\alpha - \alpha_0)$

$P = m \cdot V$ $T = 2\pi\sqrt{\frac{l}{g}}$

02

Differential Equations

Making a difference

Differential Equations

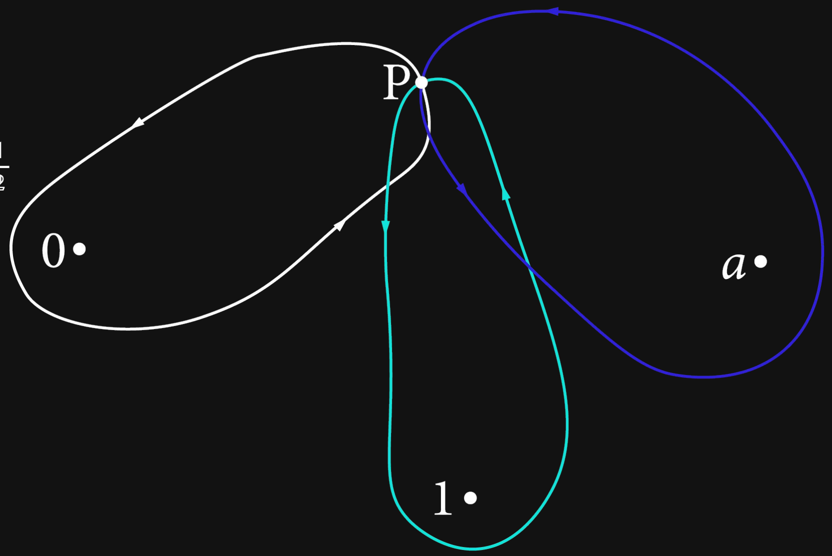
- Equations relating complex functions and their derivatives.
- Define a function on small area of the complex plane.

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0$$

The Heun Equation

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{dw}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-a)}w = 0.$$

- It has two independent solutions: called f and g
- Lamé Equation when $\gamma, \delta, \varepsilon \in \mathbb{Z} + \frac{1}{2}$
- Singular at $0, 1,$ and a
- How do the solutions change as we continue around each singularity?



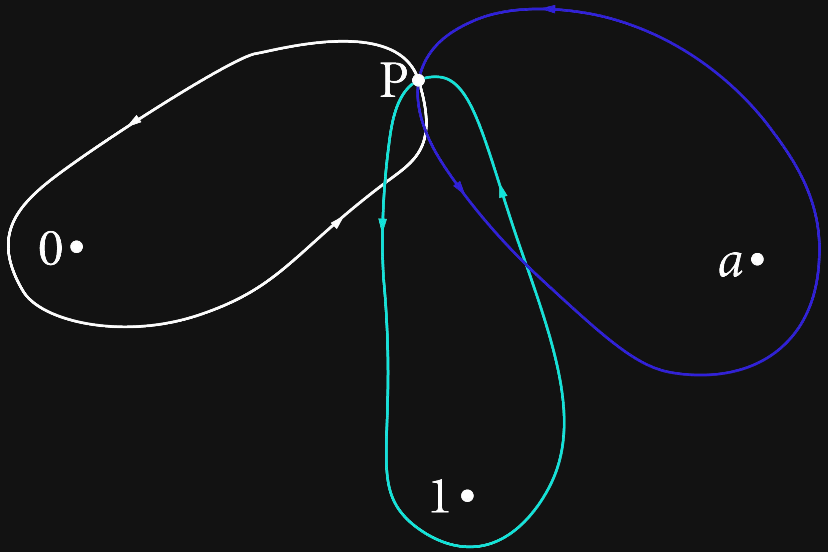
An Example

Take the solution f

As we go around 0, f morphs into a different solution, say $f + .g$

Similarly g might morph into $2f - g$.

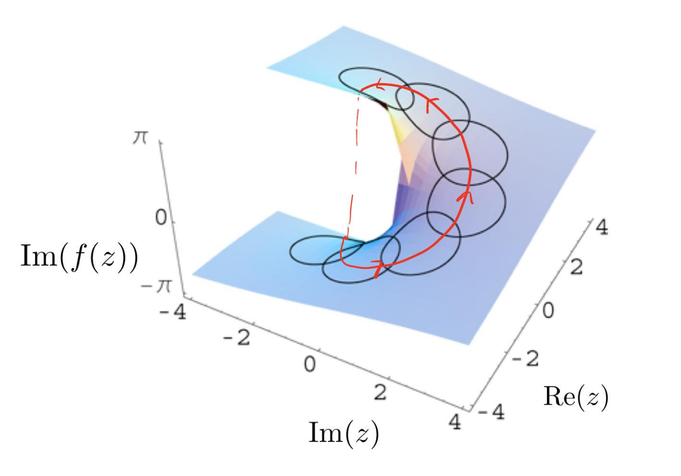
This gives us the map $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$



Monodromies

- From looping, we get three matrices $M_0,\ M_1,$ and M_a
- We refer to these maps as **monodromies** (**monodromy**, singular).

$$M_0,\ M_1,\ M_a : f \rightarrow f^*$$



03

Our Project

To make the world a better place.

Unitarity

- Define the **monodromy group** G of an equation as the group generated by its monodromy matrices (under multiplication)
- A **Hermitian matrix** H is preserved complex-conjugate transposition:

$$\overline{H^T} = H$$

- We call the group of monodromies **unitary** if, for all $g \in G$, there is an H that satisfies:

$$\overline{g^T} H g = H$$

Research Question

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{dw}{dz} + \frac{\alpha\beta z - B}{z(z-1)(z-a)}w = 0.$$

- We call this B the **accessory parameter**.
- We seek to answer: For what values B does this equation possess a unitary monodromy group?
- Our question is equivalent to: For what values B does this above equation admit real-analytic solutions?
- **Significance:** This question is important for the *analytic Langlands correspondence*, a theory attempting to link complex curves with algebraic data.

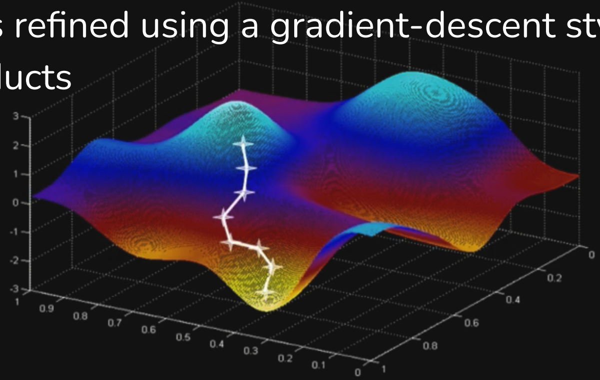
04

Analysis

Made a difference?

Algorithmic Approach

- We extend a method from (Beukers, ‘07) to compute desired values of B using computational methods. Works when **all** monodromy matrices are reflections (eigenvalues 1, -1).
- Computational algorithm:
    - Guess an initial value of B
    - Repeatedly gets refined using a gradient-descent style method and traces of matrix-pair products



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Our Heun Unitarity Theorem

Theorem. *Let G be the monodromy group generated by the matrices $P, Q, R \in GL(2, \mathbb{C})$, and assume that PQR is parabolic. Then, for the statements*

1. *G is unitary,*
2. *For $\lambda_P = e^{-\pi i \gamma}$, $\lambda_Q = e^{-\pi i \delta}$, $\lambda_R = e^{-\pi i \varepsilon}$, we have*

$$\frac{\operatorname{tr}(PQ)}{\lambda_P \lambda_Q}, \frac{\operatorname{tr}(QR)}{\lambda_Q \lambda_R}, \frac{\operatorname{tr}(PR)}{\lambda_P \lambda_R} \in \mathbb{R},$$

$(1) \implies (2)$ in general and $(2) \implies (1)$ when two of P, Q, R are reflections.

So what?

- Previously, **ALL of P, Q, R had to be reflections** (i.e eigenvalues 1, -1). In other words, it only works for the Lamé Equation
- Our theorem allows us to rigorously extend the algorithm to the case where only **TWO of P, Q, R need to be reflections** (the third matrix is free to vary). This extends our results to an infinite class of Heun Equations.
- Furthermore, our theorem shows that in all cases of the Heun Equation, our algorithm greatly restricts the possible values of B for which the monodromy group is unitary (i.e the set of values we find MUST contain the unitary values of B).

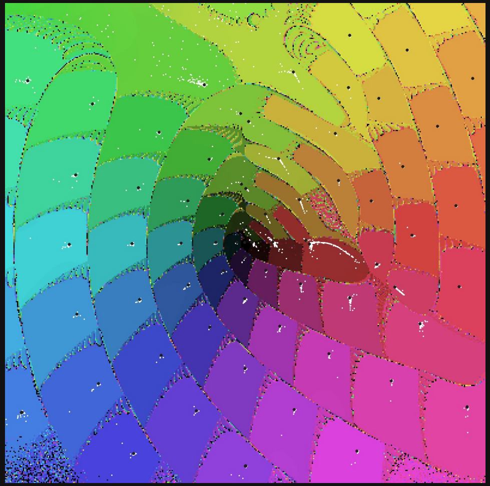
05

Numerical Results

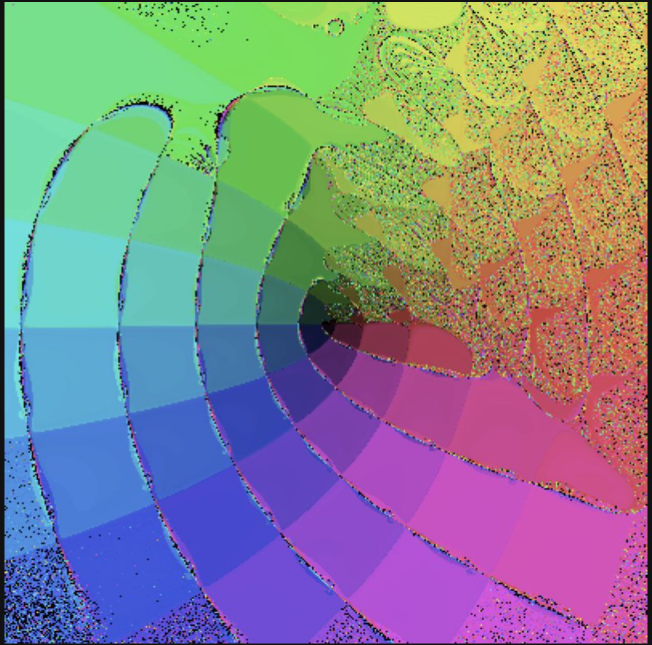
Ooh, pretty!

Results

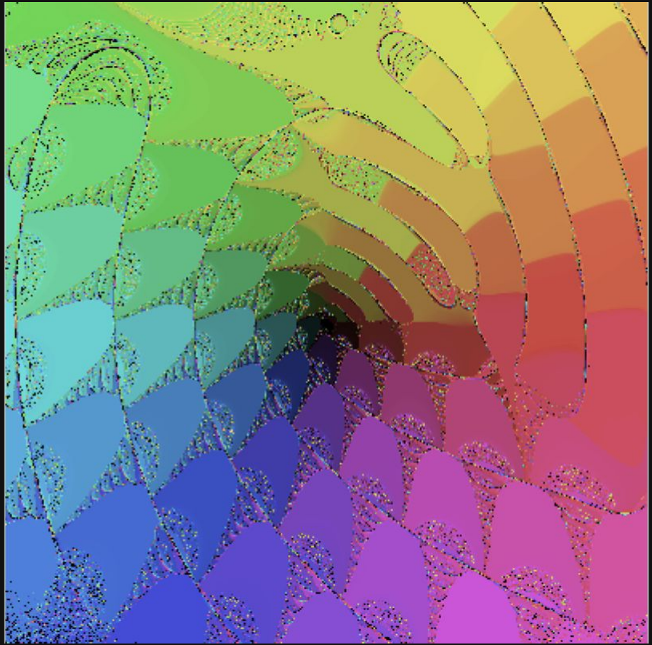
- The following colorings of the complex plane represent the outputs of our algorithm.
- The position represents the initial guess we used for B
- The coloring represents the final complex value of B (using an HSV Color Transform)
- Beautiful Spiral Patterns



More results



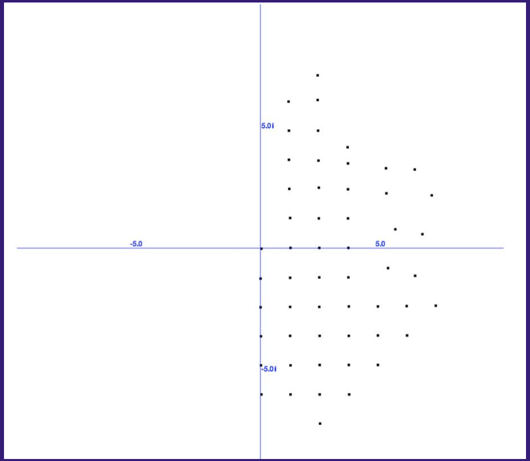
Heun Equation with ε=0.125



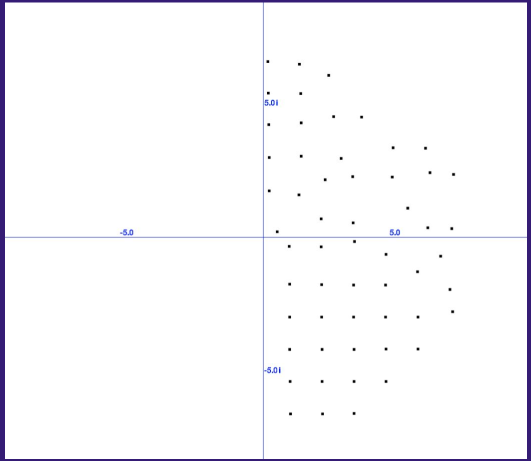
Heun Equation with ε=0.625

Lattice of Gaussian Integers

- By Beukers's predictions the unitary values should be the squares of distorted Gaussian Integers ($a + bi$, where a, b are integers)
- Indeed, taking the square roots of our numerical results, we get the following lattices:



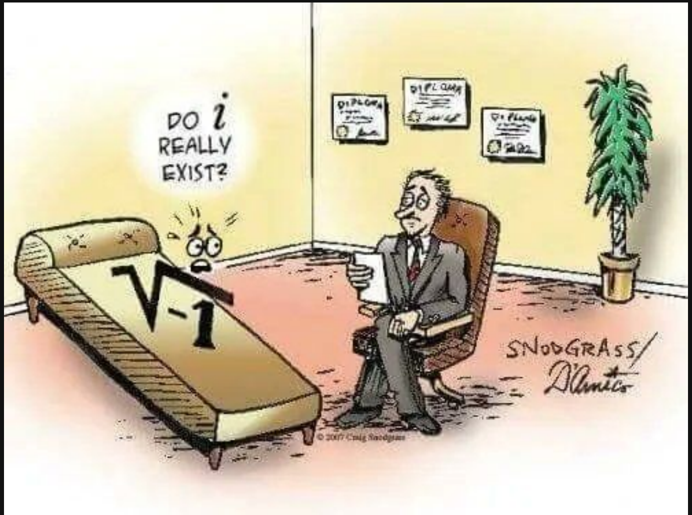
The Lamé Equation



Heun Equation with ε=0.125

Thank you for hanging in there!

We hope you enjoyed!



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Acknowledgements

We would like to thank our mentor, David Darrow, for the immense help and guidance and assistance he has provided to us throughout our entire research process.

We would like to thank Dr. Pavel Etingof for proposing our topic of study and for his insights while advising our project.

We would like to thank the MIT PRIMES program for facilitating our research.

Thanks to SlidesGo for the beautiful slides template: https://slidesgo.com/

Bibliography

Frits Beukers. Unitary monodromy of Lamé differential operators. Regular and Chaotic Dynamics, 12(6):630–641, 2007.

(*Beukers (2007) has worked on this problem for a specific version of our equation (the Lamé Equation)*)