On the Uniqueness of Certain Types of Circle Packings on Translation Surfaces

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MIT PRIMES-USA

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Overview

What is a translation surface?

- Folding a square.

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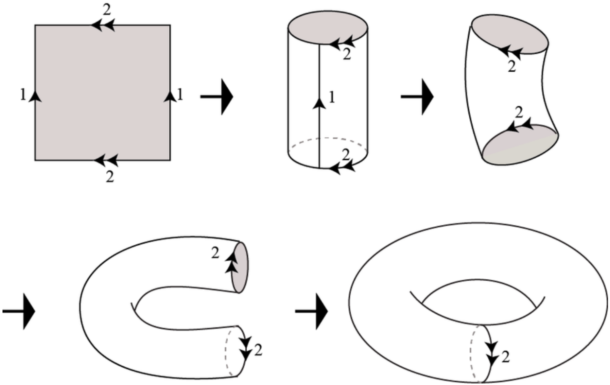
- Folding a square.



What is a translation surface?

- Folding a hexagon.

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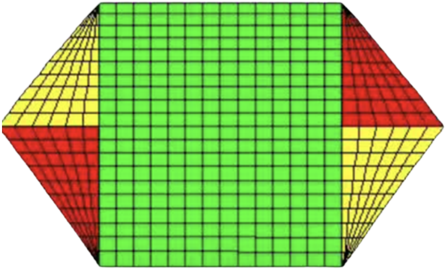

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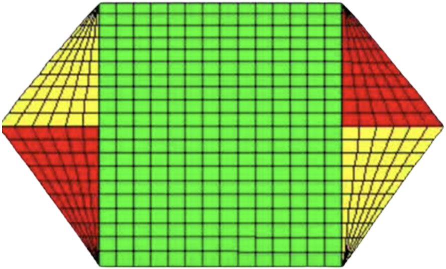
- Folding a hexagon.



- Animation Link

What is a translation surface?

- Folding an octagon.

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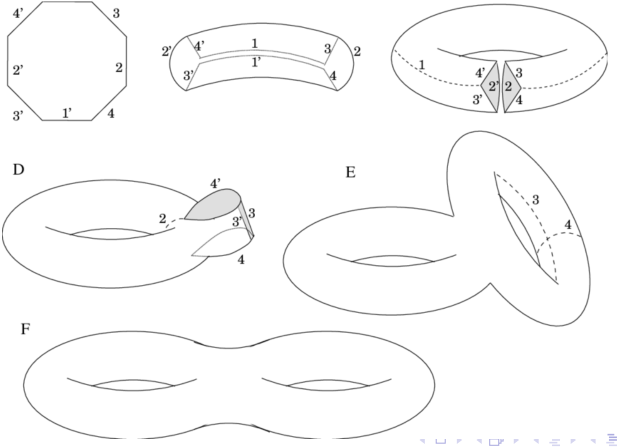

What is a translation surface?

- Start with a polygon that has an even number of sides.
- Opposite sides are parallel and of equal length.
- Identify opposite sides together and fold along them successively.

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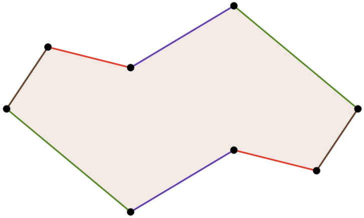

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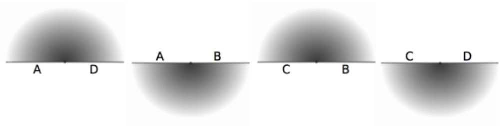
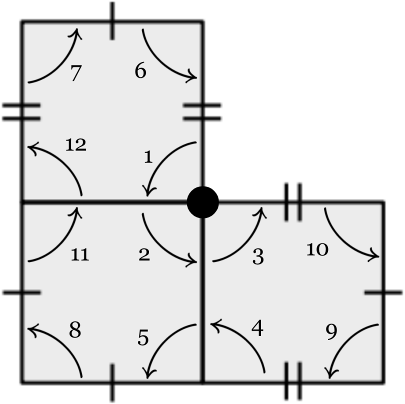
- Angle at cone point of the form $2\pi \cdot (k+1)$ for some $k > 0$.
- Neighborhood around a cone point is isometric to neighborhood around the origin in the following diagram:



Example of a Cone Point



Degrees and Strata

- Suppose that the n cone points have degrees $d_1, d_2, \cdots, d_n$. Then:

$$\sum_{i=1}^{n} d_i = 2g - 2$$

where $g > 1$ is the genus of the translation surface.

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  where $g > 1$ is the genus of the translation surface.
- Let $g > 1$ and consider a partition κ of $2g - 2$. We define a *stratum* $\mathcal{H}(\kappa)$ to be a collection of translation surfaces such that the order of each cone point is given by κ.

Genus Two Strata

- When $g = 2$, we have two possible cases.
- One cone point of degree 2, denoted $\mathcal{H}(2)$ or two cone points of degree 1, denoted $\mathcal{H}(1,1)$.

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- One cone point of degree 2, denoted $\mathcal{H}(2)$ or two cone points of degree 1, denoted $\mathcal{H}(1,1)$.
- Every translation surface M of genus 2 is hyperelliptic (i.e. admits a conformal involution $\eta : M \to M$ with exactly six fixed points).

Doubled Slit Torus

Theorem (McMullen, 2007)

Let M be a translation surface of genus 2. Then M contains a geodesic J such that $J \neq \eta(J)$ and splits along $J \cup \eta(J)$ into the connected sum of two slit tori.

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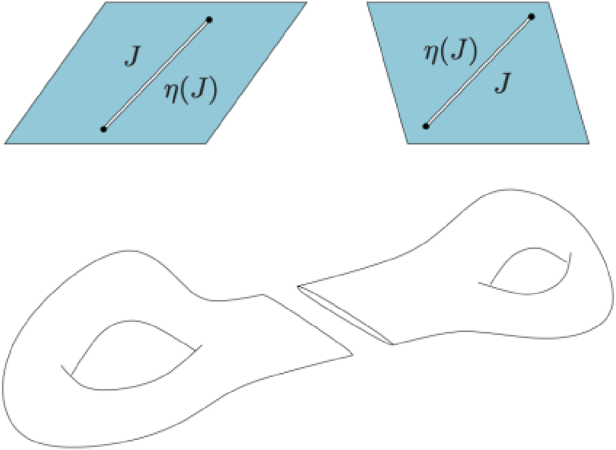

Triangulations

- A triangulation of a surface S is a locally finite decomposition of S into a collection of topologically closed triangles such that any two either:
  - are entirely disjoint
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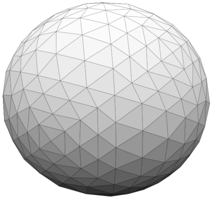

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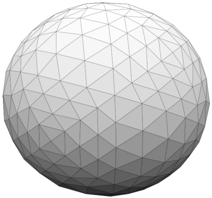

- Triangulations are allowed to be degenerate (loops and bigons).

Contacts Graph

- A *contacts graph* is a graph with n vertices $v_1, v_2, \ldots, v_n$ corresponding to the generalized circles $c_1, c_2, \ldots, c_n$ such that v_i and v_j are connected if and only if c_i and c_j are externally tangent

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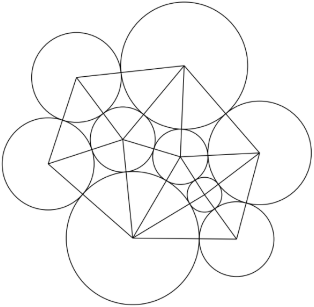
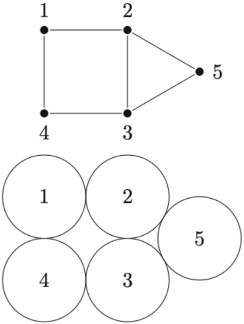
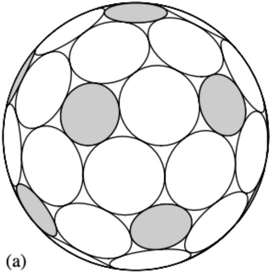




Circle Packing

- A *circle packing* is a configuration of generalized circles on the surface such that the contacts graph is a triangulation.



(a)

Circle Packing Theorem

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Theorem (Koebe-Andreev-Thurston)

- *Let K be a simple planar graph.*
- *Then there exists a collection of topological circles $\mathcal{P}_K$ on the Riemann sphere with K as its contacts graph.*
- *This circle configuration is univalent and unique (up to the Möbius transformation).*

Guiding Questions

- For a given triangulation of a translation surface in $\mathcal{H}(1,1)$, are circle packings unique up to the hyperelliptic involution?

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- For a given triangulation of a translation surface in $\mathcal{H}(1,1)$, are circle packings unique up to the hyperelliptic involution?
- Given an arbitrary triangulation T of a genus 2 translation surface M, can one always find a circle packing of some M' with contacts graph T such that M and M' lie in the same stratum?

Our Work

Theorem

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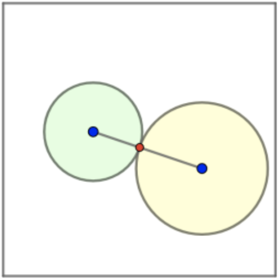
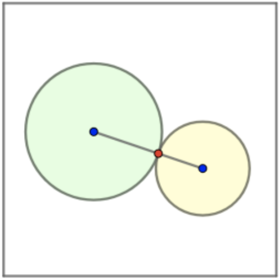
- *Suppose that the packing contains two externally tangent double circles C_1 and C_2 such that the slit connects the centers of the two circles.*
- *If C_1 and C_2 are fixed in place on the doubled slit torus, the packing can vary in only finitely many ways.*

Diagram





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- My parents

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