The Probabilistic Method

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Introduction

Probabilistic Method (Ver. 1)

Try to prove that a structure with certain desired properties exists by showing the properties hold with positive probability.

"$P(\text{Event}) > 0 \implies$ Event is non-empty."

Definition (Expected Value)

The expected value of a random variable with possible outcomes x is given by

$$E[X] = \sum_x xP[X = x].$$

Probabilistic Method (Ver. 2)

If X is a random variable, there exists a point in the probability space for which $X \geq E[X]$ and a point for which $X \leq E[X]$.

"$\exists$ value greater than or equal to the mean."

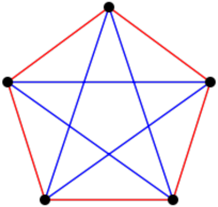
Ramsey Numbers

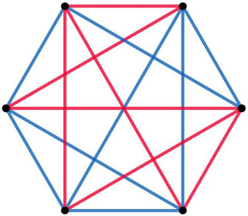
Definition (Ramsey Numbers)

The *Ramsey number* $R(k, l)$ is the smallest $n \in \mathbb{Z}$ such that in any two-coloring of the edges of a K_n graph, there is either a red K_k or a blue K_l.

Examples

$R(3,3) = 6$





Examples

	l				
k		3	4	5	6
	3	6	9	14	18
	4	9	18	25	[35, 41]
	5	14	25	[43, 49]	[58, 87]
	6	18	[35, 41]	[58, 87]	[102, 165]

Proposition

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

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k	3	4	5	6	7	8	9	10	11	12
n	3	6	11	17	27	42	65	100	152	231

Table: $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \approx 1$

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Proof.

$R :=$ subset of k vertices

$A_R :=$ event where R has only red or only blue edges

$Pr[A_R] = \frac{2}{2^{\binom{k}{2}}} \implies Pr[A] \leq \binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$

$Pr[\text{no } A_R] > 0$, so there exists some coloring that does not satisfy the conditions. $\square$

Theorem

If there exists a real $p \in [0,1]$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then $R(k,t) > n$.

Proof.

$p :=$ probability any given edge is colored red
$R :=$ subset of k vertices, $B :=$ subset of t vertices
$A_R :=$ event where R has only red edges, $B_B :=$ event where B has only blue edges
$Pr[\text{no } A_R \text{ or } B_B] > 0$, so there exists some coloring that does not satisfy the conditions. $\square$

Bounds on Ramsey Numbers

Proposition

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

Corollary 1

$R(k,k) > \lfloor 2^{k/2} \rfloor$.

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Corollary 2

$R(4,t) \geq \Omega((t/\log t)^{3/2})$.

Theorem

For any n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$.

Ramsey Numbers and Linearity of Expectation

Theorem

For any n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$.

Proof.

$R :=$ subset of k vertices
$X_R := 1$ iff K_k determined by R is monochromatic
$E[X] = \binom{n}{k} 2^{1-\binom{k}{2}} = m$
Take a two-coloring with $X \leq E[X]$
Remove one vertex from each R where $X_R = 1$
At most m vertices removed, so $s \geq n - m$ $\square$

Some More Results

Theorem

If there exists $p \in [0,1]$ with

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l} (1-p)^{\binom{l}{2}} < 1,$$

then $R(k,l) > n$.

Corollary

For all integers n and $p \in [0,1]$,

$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}.$$

The Heilbronn Triangle Problem

Let's Play a Game

You pick a set S of n points in a unit square. I pick the smallest triangle with vertices in S. You win if the triangle is big. How big can you force the triangle to be in terms of n?

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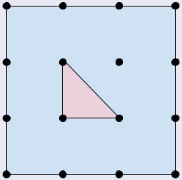
The Problem

Q: What is

$$\max_{|S|=n} \min_{\triangle \text{ in } S} \text{Area}(\triangle) := T(n)?$$

Example and Initial Conjecture

Example with $n = 16$



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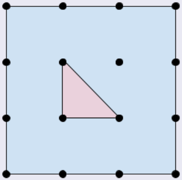


Heilbronn conjectured that $T(n) = O(1/n^2)$, an upper bound.

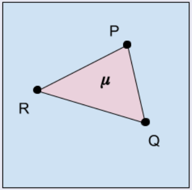
A Lower Bound

Theorem (1)

There is a set S of n points in unit square U such that $T(S) \geq 1/(100n^2)$.

A Lower Bound

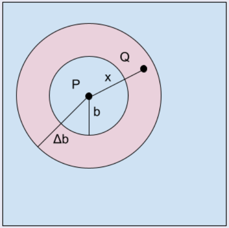
Proof.



Look for upper bound on $Pr[\mu \leq \epsilon]$.

A Lower Bound

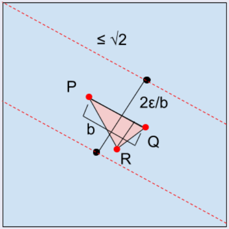
Proof.



$$Pr[b \leq x \leq b + \Delta b] \leq (b + \Delta b)^2\pi - b^2\pi = 2\pi b\Delta b + [(\Delta b)^2\pi].$$

A Lower Bound

Proof.

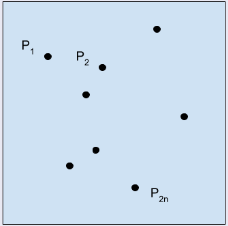


Since $0 \le b \le \sqrt{2}$, we have

$$Pr[\mu \le \epsilon] \le \int_0^{\sqrt{2}} (2\pi b)(4\sqrt{2}\epsilon/b)db = 16\pi\epsilon.$$

A Lower Bound

Proof.



Let X denote the number of triangles $P_iP_jP_k$ with area at most $1/(100n^2)$. Taking $\epsilon = 1/(100n^2)$ from above, the probability for a particular i, j, k is $16\pi/(100n^2) < 0.6/n^2$.

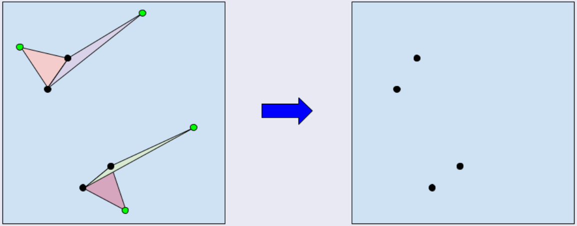
A Lower Bound

Proof.

Then

$$E[X] < \binom{2n}{3} 0.6/n^2 < n.$$

Thus there exists a specific set of $2n$ vertices with fewer than n triangles of area less than $1/(100n^2)$.



Delete one vertex from each such triangle. This leaves at least n vertices, and no triangle has area less than $1/(100n^2)$. □

Best Known bounds

Conjecture (Heilbronn)

$T(n) = O\left(\frac{1}{n^2}\right)$ *(Upper Bound).*

Theorem (1)

$T(n) = \Omega\left(\frac{1}{n^2}\right)$ *(Lower Bound).*

Theorem (Komlós, Pintz & Szemerédi (1981))

$T(n) \leq \left(\frac{\exp(c\sqrt{\log n})}{n^{8/7}}\right)$ *for some constant c (Upper Bound).*

Theorem (Komlós, Pintz & Szemerédi (1982))

$T(n) = \Omega\left(\frac{\log n}{n^2}\right)$ *(Lower Bound).*

Acknowledgements

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References I

[1] N. Alon, J.H. Spencer, *The Probabilistic Method*. New Jersey: John Wiley & Sons, Inc., 2016.