Representation Theory of the Symmetric Group

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What is Representation Theory?

In non-rigorous terms, Representation Theory is the study of representing abstract algebraic structures like groups using concrete matrix transformations. To better demonstrate this notion, we will focus on representing the *symmetric groups* for the most part.

Throughout this presentation we will be using information from [Sagan, 2013].

Symmetric Groups

Definition of Symmetric Groups

For an positive integer n, group S_n is the group of permutations of the integers $1, 2, 3, ..., n$. These permutations satisfy the property of a group: inverse, closure, identity.

A concrete example of an element $\pi \in S_5$ is:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} \rightarrow \pi(1) = 3, \pi(2) = 4, \pi(3) = 5, \pi(4) = 2, \pi(5) = 1$$

Cycle Decomposition

For the above element π we can write it as (135)(24), or alternatively as (15)(13)(24), where the portion inside a parenthesis is a *cycle*.

Group representations and a simple example

Definition.

A group representation of G is a group homomorphism:

$$\rho : G \to GL(V)$$

where $GL(V)$ is the group of all invertible linear maps from V to V and V is a complex vector space. If we fix the basis of V and $\dim V = d$, we can also say that each element $g \in G$ is assigned a $d \times d$ matrix $\rho(g)$ such that:

1. $\rho(e) = I$ for the identity $e \in G$.
2. $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.

We will focus on the case when $G = S_n$.

Trivial representation

The trivial representation maps all the elements of a group G, say S_n, to $[1]$ in $GL_1(\mathbb{C})$.

Some examples

Sign representation

Each element $\pi \in S_n$ is assigned a "sign," $\text{sgn}(\pi)$, "+1" or "-1". This "sign" depends on the times π inverts a pair of numbers in $1, 2 \ldots n$. The sign will be positive "+1" if an even number pairs are inverted, negative "-1" if an odd number pairs are inverted. Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (24)(15)(13) = (25)(15)(12)(13)(24),\ \text{sgn}(\pi) = -1$$

Remark

The sign is independent of the decomposition of π one looks at. The proof for this remark comes from the observation that any inversion of two numbers can be written as an odd number sequence of inversions.

Some examples

Defining representation

The *defining representation* of S_n comes from permuting the standard basis vectors of $GL(V)$ which is of degree n. That is, assigning elements with matrices that permute the basis vectors of $\mathbb{C}\{e_1, e_2, ..., e_n\}$, where e_i is the n by 1 vector with all 0's but a 1 on the i^{th} entry.

$$\text{For } \pi = (12)(3),\ \rho((12)) = [e_{\pi(1)}|e_{\pi(2)}|e_{\pi(3)}] = [e_2|e_1|e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{For } \pi = (132),\ \rho((132)) = [e_3|e_1|e_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Some examples

Another important representation for a group is the *regular representation*.

Regular representation

Regular representation is recovered a group G acting on itself from the left.

Example: Let $G = C_4$. Let $S = G = \{e, g, g^2, g^3\}$ whose elements will become the standard basis in a vector space. We then have $\mathbb{C}[G]$=$\{c_1\mathbf{e} + c_2\mathbf{g} + c_3\mathbf{g^2} + c_4\mathbf{g^3}$, for $c_i \in \mathbb{C}\}$. With the G group action: $g_i(c_1\mathbf{e} + ... + c_4\mathbf{g^3}) = c_1(\mathbf{g_i e}) + ... + c_4(\mathbf{g_i g^3})$.
Multiplication $g_i g_k = g_m$ yields $\mathbf{g_i g_k} = \rho(g_i)\mathbf{g_k} = \mathbf{g_m}$.

$$\rho(g) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \rho(g^k) = \rho(g)^k$$

Reducibility of representations and simple examples

Definition. (subrepresentation)

A subrepresentation W of V is a vector subspace of V that is closed under group action, so that for all $g \in G, w \in W$:

$$gw \in W$$

nonproper and zero subrepresentation

Any G-representation V has the subrepresentations $W = V$ and $W = \{0\}$, where 0 is the zero vector. Any subrepresentation that is not the whole vector space is *proper*, and $W = \{0\}$ is called a *zero* representation. We are only interested in nonzero proper subrepresentations, which are subrepresentations other than these two.

Non-trivial subrepresentations

subrepresentation of the defining representation

Let $G = S_3$, $S = \{e_1, e_2, e_3\}$, consider the $G-$ *representation* for defining representation, $V = \mathbb{C}S$. We construct the Subrepresentation of V $W = \mathbb{C}\{e_1 + e_2 + e_3\} = \{c(e_1 + e_2 + e_3), \text{ for } c \in \mathbb{C}\ \}$

For the defining representation, S_3 acts on the basis by permuting vectors in the basis. In constructing the subrepresentation W, we are summing the basis vectors of V into one single vector. W is invariant under the group action, so it is indeed a subrepresentation. For $\pi \in S_3$:

$$\pi \cdot c(e_1 + e_2 + e_3) = c(e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)}) = c(e_1 + e_2 + e_3)$$

The order of adding V basis vectors is changed but the sum is an invariant.

Complete reducibility of representations

Definition.

Suppose we have a nonzero representation X from G to $GL(V)$. Then we say V is *reducible* if it contains a non-trivial subrepresentation W. Otherwise, V is said to be *irreducible*.

Definition. *(direct sum)*

Let V be a vector space with subspaces U and W. Then V is the internal direct sum of U and W, written $V = U \oplus W$, if every $v \in V$ can be uniquely expressed as the sum:

$$v = u + w, \quad u \in U, w \in W$$

The direct sum allows us to break the vector space into smaller pieces of subspaces. If each subspace is indeed a subrepresentation, then we only need to study how G acts on these separate pieces.

Maschke's theorem

Theorem.

Let G be a finite group and V be a nonzero G-representation. Then

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where each W_i is an irreducible G-subrepresentation of V.

Since Maschke's theorem states that all G-representations can be reduced into a direct sum of irreducible representations, we can find all representations of G by studying only the irreducible representations of a finite group.

Some examples

Let's go back to the subrepresentation we have previously on the defining representation of S_3. We have $G = S_3, S = \{e_1, e_2, e_3\}$, $V = \mathbb{C}S$, and the subrepresentation $W_1 = \mathbb{C}\{e_1 + e_2 + e_3\}$. Consider the complement of W_1, $W_2 = \mathbb{C}\{e_1 - e_2, e_2 - e_3\}$, for $w = a(e_1 - e_2) + b(e_2 - e_3) \in W_2$, we have:

$$\pi w = a(e_{\pi(1)} - e_{\pi(2)}) + b(e_{\pi(2)} - e_{\pi(3)})$$

We have $w = ae_1 + be_2 + ce_3 \in W_2$ is equivalent to $a + b + c = 0$ and this is invariant under π's action. Thus, W_2 is also a subrepresentation and the defined representation of S_3 can be decomposed as:

$$V = W_1 \oplus W_2 = \mathbb{C}\{e_1 + e_2 + e_3\} \oplus \mathbb{C}\{e_1 - e_2, e_2 - e_3\}$$

Partitions

If $\lambda = (5, 2, 2, 1)$, we write $\lambda \vdash 10$ to mean that $5 + 2 + 2 + 1 = 10$ and furthermore the terms of λ are positive integers and are in nonstrict decreasing order. λ is called a *partition* of 10. In general, we write $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$, so in our example, $\lambda_1 = 5$, $\lambda_2 = 2$, etc.

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & & & \\
\bullet & \bullet & & & \\
\bullet & & & &
\end{array}$$

Above is the *Ferrers diagram* for λ, with the ith row having λ_i dots. We may replace the dots with each of the numbers 1 through 10 without repetition to get a λ-*tableau*:

$$\begin{array}{ccccc}
4 & 5 & 7 & 8 & 2 \\
10 & 6 & & & \\
1 & 9 & & & \\
3 & & & &
\end{array}$$

Young Tabloids

If we let

$$t = \begin{array}{lllll} 4 & 5 & 7 & 8 & 2 \\ 10 & 6 & & & \\ 1 & 9 & & & \\ 3 & & & & \end{array}$$

then t defines a λ-*tabloid*

$$\{t\} = \begin{array}{ccccc} \hline 4 & 5 & 7 & 8 & 2 \\ \hline 10 & 6 & & & \\ \cline{1-2} 1 & 9 & & & \\ \cline{1-2} 3 & & & & \\ \cline{1-1} \end{array}$$

The order of the numbers in any given row of a tabloid does not matter; it is the same tabloid. On the other hand, in a tableau, the order of numbers in rows does matter.

S_n action on tableaus and tabloids

If $\lambda \vdash n$, then S_n acts on the set of λ-tableaus, and similarly the set of λ-tabloids by acting on each entry. For example,

$$(1\,2)(4\,6\,3)\ \begin{matrix} 4 & 5 & 7 & 8 & 2 \\ 10 & 6 & & & \\ 1 & 9 & & & \\ 3 & & & & \end{matrix} \quad = \quad \begin{matrix} 6 & 5 & 7 & 8 & 1 \\ 10 & 3 & & & \\ 2 & 9 & & & \\ 4 & & & & \end{matrix}$$

and

$$(1\,2)(4\,6\,3)\ \begin{array}{ccccc} \hline 4 & 5 & 7 & 8 & 2 \\ \hline 10 & 6 & & & \\ \cline{1-2} 1 & 9 & & & \\ \cline{1-2} 3 & & & & \\ \cline{1-1} \end{array} \quad = \quad \begin{array}{ccccc} \hline 6 & 5 & 7 & 8 & 1 \\ \hline 10 & 3 & & & \\ \cline{1-2} 2 & 9 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1} \end{array}$$

Permutation representation corresponding to a partition

We may take formal linear combinations of λ-tabloids (i.e. elements of $\mathbb{C}\{\{t_1\},\{t_2\},\ldots,\{t_k\}\}$ where $\{t_1\},\{t_2\},\ldots,\{t_k\}$ is a complete list of distinct λ-tabloids). We may apply permutations to these formal linear combinations through linear extension, thus creating a representation of S_n with the vector space of these linear combinations; we call this representation M^{λ}; for example, in $M^{(5,2,2,1)}$,

$$
\begin{aligned}
&(1\,2)(4\,6\,3)\left(2\begin{array}{ccccc}\hline 4 & 5 & 7 & 8 & 2 \\ \hline 10 & 6 & & & \\ \cline{1-2} 1 & 9 & & & \\ \cline{1-2} 3 & & & & \\ \cline{1-1}\end{array} - 15\begin{array}{ccccc}\hline 2 & 4 & 8 & 10 & 1 \\ \hline 6 & 5 & & & \\ \cline{1-2} 3 & 2 & & & \\ \cline{1-2} 7 & & & & \\ \cline{1-1}\end{array}\right) \\
&= 2\begin{array}{ccccc}\hline 6 & 5 & 7 & 8 & 1 \\ \hline 10 & 3 & & & \\ \cline{1-2} 2 & 9 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1}\end{array} - 15\begin{array}{ccccc}\hline 1 & 6 & 8 & 10 & 2 \\ \hline 3 & 5 & & & \\ \cline{1-2} 4 & 2 & & & \\ \cline{1-2} 7 & & & & \\ \cline{1-1}\end{array}
\end{aligned}
$$

The polytabloid corresponding to a tableau

Definition *(polytabloid corresponding to tableau)*

Let t be a tableau. We define C_t to be the set of permutations π such that each column of πt has the same set of numbers to the corresponding column of t. Then the *polytabloid corresponding to t* is

$$\mathbf{e_t} = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\{\pi \mathbf{t}\}$$

For example, if $t = \begin{array}{ccc} 2 & 4 & 1 \\ 3 & 5 & \end{array}$, then

$$\mathbf{e_t} = \begin{array}{ccc} \hline 2 & 4 & 1 \\ \hline 3 & 5 & \\ \hline \end{array} - \begin{array}{ccc} \hline 3 & 4 & 1 \\ \hline 2 & 5 & \\ \hline \end{array} - \begin{array}{ccc} \hline 2 & 5 & 1 \\ \hline 3 & 4 & \\ \hline \end{array} + \begin{array}{ccc} \hline 3 & 5 & 1 \\ \hline 2 & 4 & \\ \hline \end{array}$$

Definition *(Specht modules)*

If $\lambda \vdash n$, then the corresponding *Specht module* S^λ is defined by the span of the $\mathbf{e_t}$.

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Proposition

The Specht modules form a full list of irreducible representations of S_n. We also have $S^\lambda \cong S^\mu$ if and only if $\lambda = \mu$, so the list has no repetition. Furthermore, S^λ has multiplicity 1 in M^λ.

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