Irreducible Characters for Verma Modules for the Orthosymplectic Lie Superalgebra $\mathfrak{osp}(3|4)$

Honglin Zhu
Mentor: Arun S. Kannan

Phillips Exeter Academy

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1. For all $x, y \in L$, $[x, y] = -[y, x]$.
2. For all $x, y, z \in L$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Some examples

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2. The set of endomorphisms on a finite dimensional vector space V (linear maps from V to itself), $\mathrm{End}(V)$, can be made into a Lie algebra, known as the general linear algebra $\mathfrak{gl}(V)$, with

$$[x, y] := xy - yx.$$

General linear algebra

Definition

We can identify the general linear algebra $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices. And the Lie bracket is defined in terms of matrix multiplication.

Representation

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Definition

A *representation* of a Lie algebra L is a Lie algebra homomorphism (a linear map that also preserves the bracket structure) $\phi : L \to \mathfrak{gl}(V)$, where V is some vector space. When the map is clear, we usually call V the representation. We also call V an *L-module*.

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2. The adjoint representation L of L:

$$\mathrm{ad} : L \to \mathfrak{gl}(L), x \mapsto [x, \cdot].$$

Semisimple Lie algebras

There is a certain type of Lie algebras called semisimple Lie algebras. Their representations have nice properties and it is an important topic in representation theory to study them.

Verma modules

Indexed by *weights* (denoted M_λ for the Verma module indexed by λ). Irreducible modules can be constructed from Verma modules. Denoted by L_λ the irreducible module of highest weight λ.

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Jordan-Hölder series

For a Verma module M_λ, there exists a sequence of submodules

$$M_\lambda = N_k \supset N_{k-1} \supset \cdots \supset N_1 \supset N_0 = 0$$

such that each quotient N_i / N_{i-1} is an irreducible module. We denote by $[M_\lambda : L_\mu]$ the multiplicity of L_μ in the sequence.

Verma modules

The Jordan-Hölder multiplicities of Verma modules of semisimple Lie algebras are controlled by the Weyl group.

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Verma modules can be similarly defined for basic Lie superalgebras, but their Jordan-Hölder multiplicities are not so well understood.

General linear Lie superalgebra

Definition

Let $V = \mathbb{C}^{k|l} = \mathbb{C}^k \oplus \mathbb{C}^l$. The Lie superalgebra $\mathfrak{gl}(k|l)$ is the set of $(k+l) \times (k+l)$ supermatrices

$$\begin{pmatrix} \text{even } k \times k & \text{odd } k \times l \\ \text{odd } l \times k & \text{even } l \times l \end{pmatrix}$$

with the superbracket defined as

$$[x, y] = xy - (-1)^{|x||y|} yx,$$

for homogeneous $x, y \in \mathfrak{gl}(k|l)$.

$\mathfrak{osp}(3|4)$

The Lie superalgebra of interest is the orthosymplectic Lie superalgebra $\mathfrak{osp}(3|4)$, whose even part is $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(4) \oplus \mathfrak{so}(3)$.

The problem

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We focus on the atypical case and calculate the Jordan-Hölder multiplicities for the Verma modules of $\mathfrak{osp}(3|4)$.

Projective modules

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Standard filtration

For each projective module P_λ, there exists a sequence of submodules

$$P_\lambda = N_k \supset N_{k-1} \supset \cdots \supset N_1 \supset N_0 = 0$$

such that each quotient N_i/N_{i-1} is a Verma module. We denote by $(P_\lambda : M_\mu)$ the multiplicity of M_μ in the sequence.

BGG reciprocity

Theorem (BGG reciprocity)

For weights $\lambda, \mu \in \mathfrak{h}^*$, we have

$$(P_\lambda : M_\mu) = [M_\mu : L_\lambda].$$

Our Strategy

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If $P_\mu = M_{\mu_1} + M_{\mu_2} + \cdots + M_{\mu_n}$ and V has weights $\nu_1, \nu_2, \ldots, \nu_k$, then

$$P_\mu \otimes V = \sum M_{\mu_i + \nu_j}.$$

Results

Theorem

With some exceptions, all Verma Modules of atypical integral highest weight have Jordan-Hölder series given by

$$M_\lambda = \sum_{\sigma\lambda \preceq \lambda} \left(L_{\sigma\lambda} + L_{\sigma\lambda-\alpha} + L_{\sigma\lambda-\alpha-\beta}\right).$$

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