Asymptotics for Iterating the Lusztig-Vogan Bijection for GL_n on Dominant Weights

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Lusztig-Vogan Bijection

Lusztig-Vogan Bijection is a correspondence between **dominant weights of reductive algebraic groups** and **vector bundles on nipotent orbits**.

Lusztig-Vogan Bijection

Background

- Lusztig (1989) and Vogan (2000) : constructed the bijection
- Bezrukavnikov (2003) : proof
- Achar (2011) : proposed algorithm
- Rush (2017) : simplified Achar's algorithm on GL_n (Type A)

Lusztig-Vogan Bijection Type A

Definition

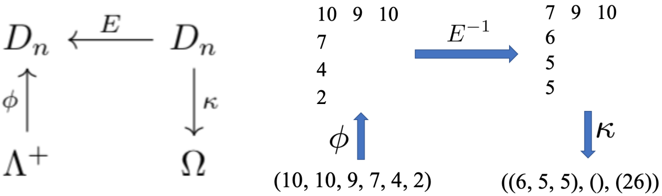

Figure – Lusztig-Vogan Bijection Type A

ϕ : construct a **weighted diagram** from a weakly decreasing sequence of integers.
E : modify a weighted diagram column-by-column by elements' **order**.
κ : Construct a set of weakly decreasing sequences of integers by **row-length**.

Lusztig-Vogan Bijection Type A

In Iteration

- Start with a weakly decreasing sequence σ
- First iteration, output is a list of weakly decreasing sequences $(\mu_1, \mu_2, \cdots, \mu_l)$
- Apply map on each list μ_i with length at least 2 and obtain a list of sequences for each.
- The procedure terminates when the length of each sequence is 0 or 1.

Example

$$(10, 10, 9, 7, 4, 2) \to ((6, 5, 5), (), (26)) \to (((), (), (16)), (), (26))$$

Lusztig-Vogan Bijection Type A

time

Definition

Let σ be a sequence of weakly decreasing integers. Then $t(\sigma)$ (**time** of σ) denotes the number of iterations needed to perform on σ until each sequence has length 0 or 1.

Example

$$(10,10,9,7,4,2) \to ((6,5,5),(),(26)) \to (((),(),(16)),(),(26))$$

$$t((10,10,9,7,4,2)) = 2$$

Our Results

Length 2

For any $x > y$, if

- $x - y \geq 2$, $LV((x, y)) = (x - 1, y + 1)$
- $0 \leq x - y \leq 1$, $LV((x, y)) = ((), (x + y))$

Example

$$(10, 3) \to (9, 4) \to (8, 5) \to (7, 6) \to ((), (13))$$

Theorem

For all weakly decreasing sequence of length 2 (x, y), we have $t(x, y) = \left\lfloor \frac{d}{2} \right\rfloor + 1$, where $d = x - y$. In addition, the final output is always $((), (x + y))$.

Our Results

Length 3

Examples

$$(10,7,0) \to (8,7,2) \to ((3),(14))$$
$$(10,4,-1) \to (8,4,1) \to (6,4,3) \to ((5),(8))$$

Theorem

For all weakly decreasing sequence of length 3 (x,y,z), *we have* $t(x,y) = \left\lfloor \frac{\min(d_1,d_2)}{2} \right\rfloor + 1$, *where* $d_1 = x - y, d_2 = y - z$.

Our Results

Length 4

Examples

$$(10, 8, 6, -5) \to (7, 7, 7, -2) \to ((-1), (), (20))$$

$$(10, 9, 8, -5) \to ((8, -3), (17)) \to ((7, -2), (17)) \to ((6, -1), (17))$$
$$\to ((5, 0), (17)) \to ((4, 1), (17)) \to ((3, 2), (17)) \to (((), (5)), (17))$$

Our Results

Length 4

Theorem

Let $\sigma = (x, y, z, w)$ be a weakly decreasing sequence. Assume that $d_1 = x - y, d_2 = y - z, d_3 = z - w, f_1 = \left\lfloor \frac{d_1}{2} \right\rfloor, f_2 = \left\lfloor \frac{d_2}{2} \right\rfloor, f_3 = \left\lfloor \frac{d_3}{2} \right\rfloor$. Then,

$$t(\sigma) = \begin{cases} \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_3 < f_1, f_2; \\ -\left\lfloor \frac{d_2}{2} \right\rfloor + \left\lfloor \frac{d_1 + d_3 + Mod(d_2, 2)}{2} \right\rfloor \text{ if } f_2 < f_1, f_3; \\ \left\lfloor \frac{d_3}{2} \right\rfloor + 1 \text{ if } f_1 < f_2, f_3; \\ \left\lfloor \frac{d_2}{2} \right\rfloor + 1 \text{ if } f_1 > f_2 = f_3 \text{ and } 2 \mid d_2 d_3; \\ \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_1 > f_2 = f_3 \text{ and } 2 \nmid d_2 d_3; \\ d_2 - \left\lfloor \frac{d_1}{2} \right\rfloor \text{ if } f_2 > f_1 = f_3; \\ \left\lfloor \frac{d_1}{2} \right\rfloor + 1 \text{ if } f_3 > f_2 = f_1 \text{ and } 2 \mid d_1 d_2; \\ \left\lfloor \frac{d_3}{2} \right\rfloor + 1 \text{ if } f_3 > f_2 = f_1 \text{ and } 2 \nmid d_1 d_2; \\ \left\lfloor \frac{\min(d_1, d_2, d_3) + 1}{2} \right\rfloor + 1 \text{ if } f_1 = f_2 = f_3. \end{cases}$$

Average Time

Definition

Definition

Let $n \geq 0, k > 0$ be integers. Let $S_{n,k}$ be the set of all length k weakly decreasing sequence whose first (largest) term is n and whose last (smallest) term is 0. Define

$$avg_k(x) := \frac{\sum_{y_{x,k}} t(y)}{|S_{x,k}|}.$$

Our Results

Average Time

Corollary

$avg_2(x) = \left\lfloor \frac{x}{2} \right\rfloor + 1$.

Corollary

$\lim_{x \to \infty} avg_3(x) = \frac{x+6}{8}$.

Theorem

$avg_4(x), 2 \mid x$ and $avg_4, 2 \nmid x$ approach two lines, both of which have slope $\frac{5}{18}$.

Our Results

Average Time

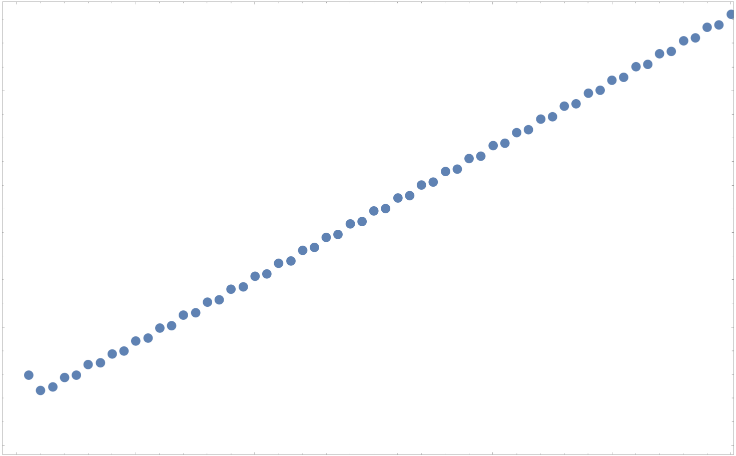

Figure – $\text{avg}_4(x)$

Our Results

Average Time

Theorem (Main)

For any positive integer $k \geq 4$, the asymptotic behavior of $avg_k(x)$, $2 \mid x$ and $2 \nmid x$ approaches two lines, both of which have the same slope.

Definition

Denote c_n as the slope of the asymptote of $avg_n(x), 2 \mid x$.

Our Results

Slope of Asymptotes

Theorem (Main)

Let $c_{2,1} = c_2, c_{3,1} = c_{3,2} = c_3$. *For* $n \geq 4$, *we define a sequence* $\{c_{n,1}, c_{n,2}, \cdots, c_{n,n-2}, c_{n,n-1}\}$ *recursively :*

$$c_{n,1} = c_{n,n-1} = \frac{n-3}{n-1}c_{n-2} + \frac{1}{2(n-1)^2}$$

$$c_{n,k} = \frac{n-2}{n-1}\left(\frac{n-2}{n-3}c_{n-2} - \frac{1}{n-3}c_{n-2,k-1}\right) + \frac{1}{2(n-1)^2} \text{ for any } 2 \leq k \leq n-2$$

Then we have $c_n = \frac{1}{n-1}\sum_{i=1}^{n-1} c_{n,i}$.

Our Results

Slope of Asymptotes

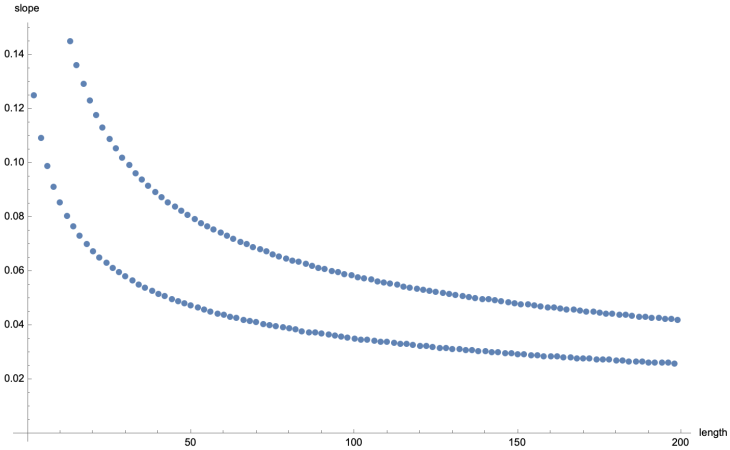

Figure – First 200 Slopes of Asymptotes of $avg_n ; c_n, 2 \leq n \leq 200$.

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Reference

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