On the Generational Behavior of Gaussian Binomial Coefficients at Root of Unity

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17.-18. October 2020

$\nu_p(n)$ denotes the number of prime factor p of n.

$$\nu_3(4) = 0; \nu_2(6) = 1; \nu_5(125) = 3.$$

Binomial Coefficients $\binom{m}{n}$ denote the number of ways to choose n objects out of m objects.

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1$$

Introduction

p-adic Valuation and Pascal Triangle

$$\nu_3(\binom{4}{0}) = \nu_3(1) = 0$$

$$\nu_3(\binom{4}{1}) = \nu_3(4) = 0$$

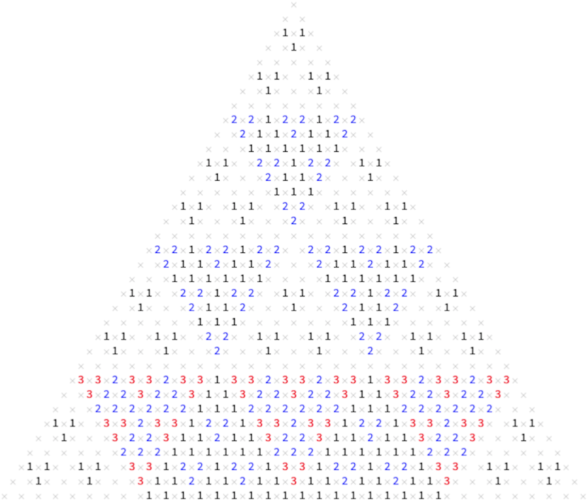
$$\nu_3(\binom{4}{2}) = \nu_3(6) = 1$$

$$\nu_3(\binom{4}{3}) = \nu_3(4) = 0$$

$$\nu_3(\binom{4}{4}) = \nu_3(1) = 0$$

Introduction

Generational Behavior of Pascal Triangle



v-Analog

When we consider the characters of quantum group, a generalization to the binomial coefficients gives the Gaussian binomial coefficients.

Definition

For $v \in \mathbb{C}, N \in \mathbb{Z}$,

$$[N]_v! = \prod_{s=1}^{N} \frac{v^s - v^{-s}}{v - v^{-1}}.$$

For $m, n \in \mathbb{Z}^*$, define the **Gaussian binomial coefficients**

$$\begin{bmatrix} m \\ n \end{bmatrix}_v = \frac{[m]_v!}{[n]_v! \cdot [m-n]_v!}.$$

It is useful in symmetry polynomials, partitions, representation theory, etc.

v-Analog

Examples

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 = \frac{2^2 - 2^{-2}}{2 - 2^{-1}} = 2 + 2^{-1} = \frac{5}{2};$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_i = \frac{(i^4 - i^{-4})(i^3 - i^{-3})}{(i^2 - i^{-2})(i - i^{-1})} = (i^2 + i^{-2})(i^2 + 1 + i^{-2}) = 0 \times 1 = 0,$$

where $i = \sqrt{-1}$.

Major Object

Analog to Pascal triangle: v-Pascal triangle.

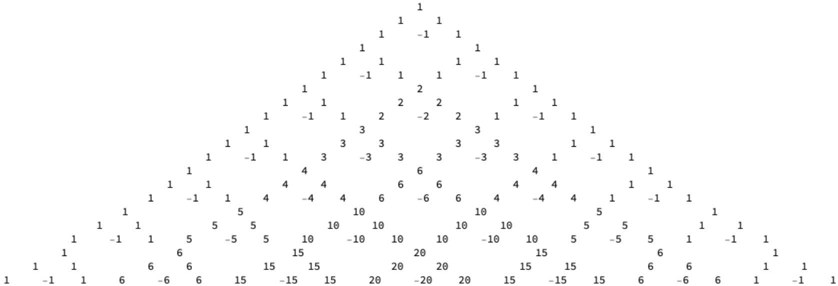


Figure: $e^{2\pi i/3}$-Pascal triangle

Representation theory background:

generational relationship between *reductive algebraic groups in prime characteristic* and *quantum groups at roots of unity*.

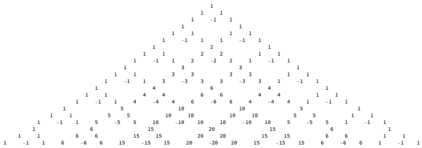
Describing v-Pascal Triangle: an elementary shadow.

v-Analog

Motivation

v-Pascal Triangle ⇒ Pascal Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



Layering Behavior

[Lusztig 1989]

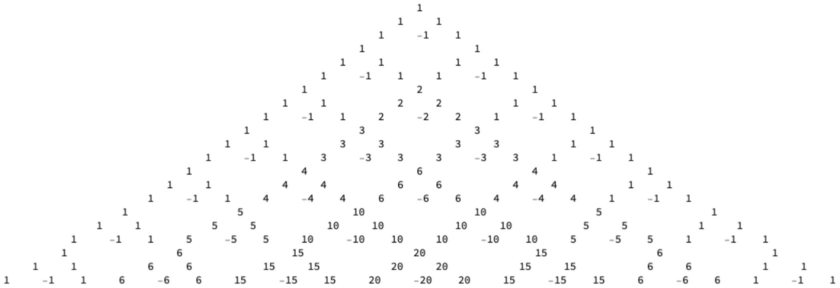


Figure: $e^{2\pi i/3}$-Pascal triangle

Integral Values

Note that:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{e^{2\pi i/5}} = e^{2\pi i/5} + e^{-2\pi i/5} = \frac{\sqrt{5}-1}{2}.$$

Pathways to integers:

Summing up all the primitive roots of unity

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^{\bullet} = \sum_{\gcd(k,q)=1, 1 \le k \le q-1} \begin{bmatrix} m \\ n \end{bmatrix}_{e^{2\pi i \frac{k}{q}}},$$

Summing up all the roots of unity

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^{\dagger} = \sum_{k=0}^{q-1} \begin{bmatrix} m \\ n \end{bmatrix}_{e^{2\pi i \frac{k}{q}}}.$$

Generational Behavior

Example: Consider 9^{th} roots of unity.

$$\begin{bmatrix} m \\ n \end{bmatrix}_9^{\bullet} = \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{2\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{4\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{8\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{10\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{14\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{16\pi}{9}}}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_9^{\dagger} = \begin{bmatrix} m \\ n \end{bmatrix}_1 + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{2\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{4\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{6\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{8\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{10\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{12\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{14\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{16\pi}{9}}}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_9^{\bullet} = -6; \begin{bmatrix} 4 \\ 1 \end{bmatrix}_9^{\dagger} = 0$$

Generational Behavior

Which one will work?

Figure: Pascal Triangle and $e^{2\pi i/3}$-Pascal Triangle 3 Valuations

The "Generational" Philosophy

Lusztig [1989, 2015]: characters of quantum groups.

Williamson [2020]: Quantum groups $\rightarrow$ algebraic groups

Lusztig & Wiliamson [2018]: Tilting characters

Elementary Version: Complication (Gaussian Binomial Coefficients) $\rightarrow$ simplicity (Binomial Coefficients)

Generational Behavior

Which one will work?

How about the p^k cases?

Original Values? Zeros? mod p? p Valuation?

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^{\bullet} or \begin{bmatrix} m \\ n \end{bmatrix}_q^{\dagger} ?$$

Solution:
Summing up all the non-one roots of unity:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^{*} = \sum_{k=1}^{q-1} \begin{bmatrix} m \\ n \end{bmatrix}_{e^{2\pi i \frac{k}{q}}}$$

A Clever Way

Summing up all the non-one roots of unity:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q^* = \sum_{k=1}^{q-1} \begin{bmatrix} m \\ n \end{bmatrix}_{e^{2\pi i \frac{k}{q}}}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_9^* = \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{2\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{4\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{6\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{8\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{10\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{12\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{14\pi}{9}}} + \begin{bmatrix} m \\ n \end{bmatrix}_{e^{\frac{16\pi}{9}}}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_9^\bullet = -6; \begin{bmatrix} 4 \\ 1 \end{bmatrix}_9^\dagger = 0; \begin{bmatrix} 4 \\ 1 \end{bmatrix}_9^* = -4$$

A Clever Way

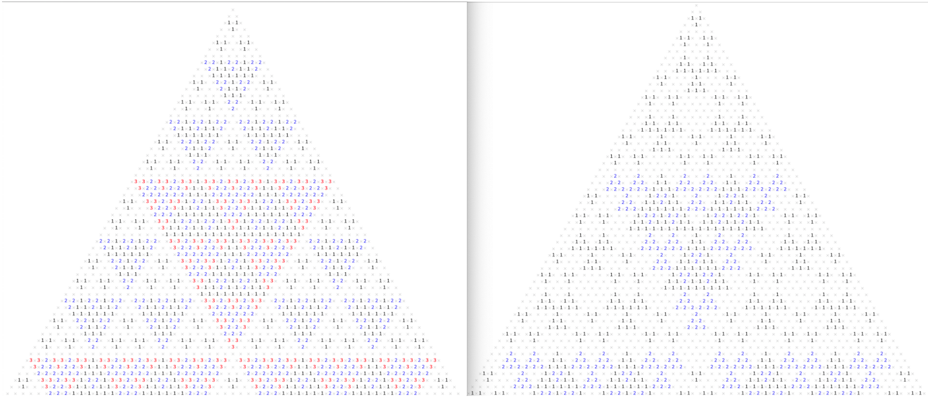


Figure: Pascal Triangle and $\begin{bmatrix} m \\ n \end{bmatrix}_9^*$ 3 Valuations

A Clever Way

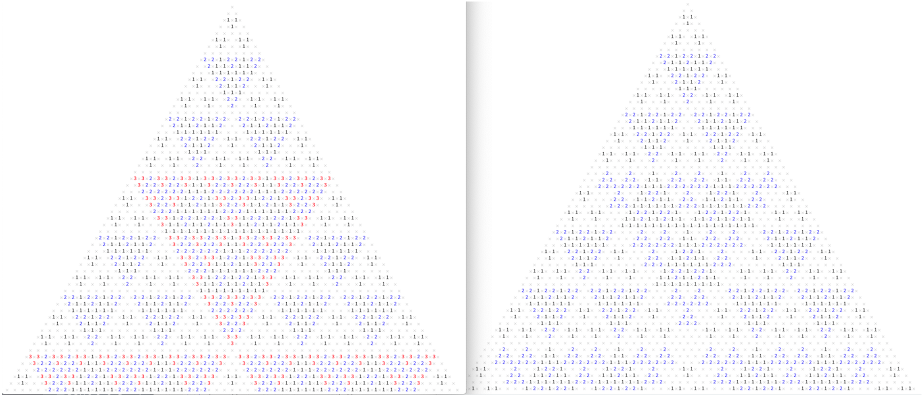


Figure: Pascal Triangle and $\begin{bmatrix} m \\ n \end{bmatrix}^*_{27}$ 3 Valuations

Formulation and Generalization

Definition

Fix an odd prime p. For a nonnegative integer k, we define the **kth generation vanishing** in the mod p Pascal's triangle to be the set of pairs (m, n) for which there are carriers the last k digits of its base p expansion, which is equivalent to

$$\binom{p^k + m \pmod{p^k}}{n \pmod{p^k}} \equiv 0 \pmod{p^k}.$$

Theorem (Main)

Fix an odd prime p. The kth generation vanishing in the mod p Pascal's triangle corresponds exactly to the vanishing of $\binom{m}{n}^*_{p^k}$*; in other words,* $\binom{m}{n}^*_{p^k}$ *vanishes if and only if* (m, n) *belongs to the kth generation vanishing of the mod p Pascal's triangle. Furthermore, if* (m, n) *does not belong to the kth generation vanishing, then*

$$v_p\binom{m}{n}^*_{p^k} = v_p\binom{m}{n}.$$

Generalization:
If we replace $\binom{m}{n}^*_{p^k}$ by $\binom{m}{n}^*_{2p^k} - \binom{m}{n}^*_{p^k} - \binom{m}{n}_{-1}$, the same result holds.

More Results

No generational behaviors present in $\begin{bmatrix} m \\ n \end{bmatrix}_q^{\bullet}$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q^{\dagger}$. However, some information is known:

$\begin{bmatrix} m \\ n \end{bmatrix}_q^{\dagger}$: Zeros and lower bounds of p valuations;
$\begin{bmatrix} m \\ n \end{bmatrix}_q^{\bullet}$: Some zeros and conjectures on zeros and signs.

Acknowledgements

We would like to thank:

- Mr. Calder Oakes Morton-Ferguson for mentorship,
- Mr. Qiusheng Li for discussion and advice all through,
- USA-PRIMES for this research opportunity,
- Yizhen Chen for proofreading our result.

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