Upho Posets

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Preliminaries

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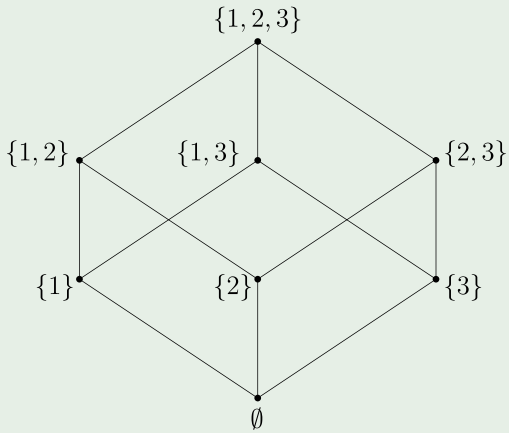

Preliminaries

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A poset P is **ranked** if $P = P_0 \sqcup P_1 \sqcup P_2 \sqcup \ldots$, such that if $x \in P_i$ and $x \lessdot y$, then $y \in P_{i+1}$.

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Definition

The **rank-generating function** of a ranked poset P is

$$F_P(x) = |P_0| + |P_1|x + |P_2|x^2 + \ldots = \sum_{k=0}^{\infty} |P_k|x^k.$$

Example

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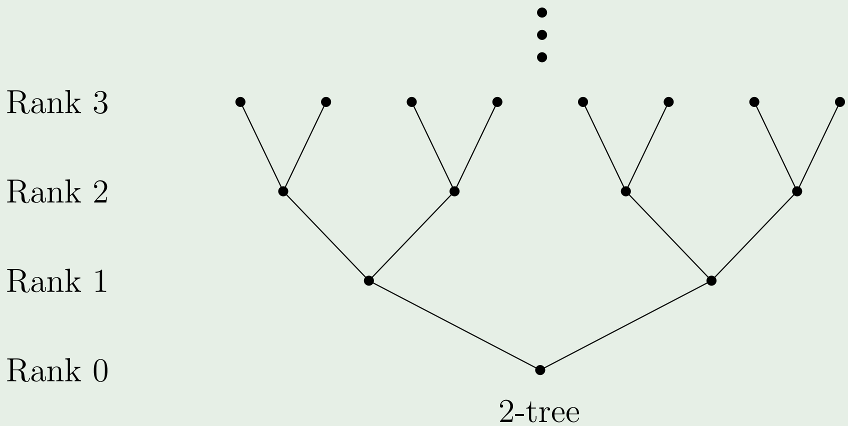
A 2-tree has rank-generating function $1 + 2x + 4x^2 + 8x^3 + \ldots = \frac{1}{1-2x}$.



The Upper Principal Order Filter

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The **upper principal order filter** above an element $s \in P$ is "s and everything above it": $V_{P,s} := \{t \in P \mid t \geq s\}$.

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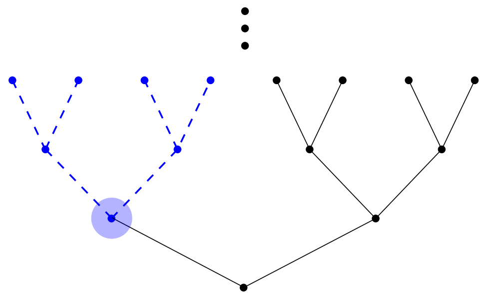
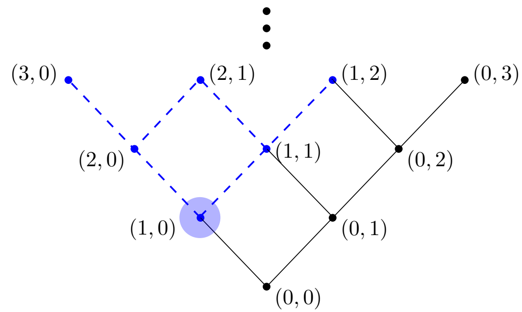
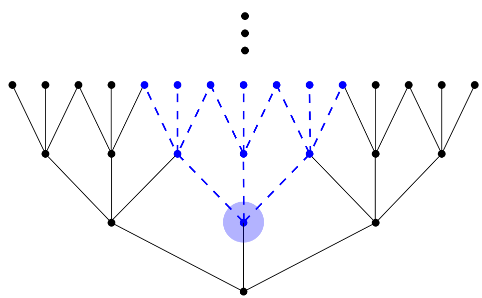


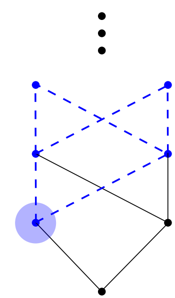
Full Binary Tree



Poset consisting of 2-dimensional Cartesian coordinates



The Stern Poset



"Bowtie" Poset

Defining Upho Posets

Definition (Stanley, 2020)

A poset P is **upho** if the upper principal order filter $V_{P,s} \cong P$ for all $s \in P$.

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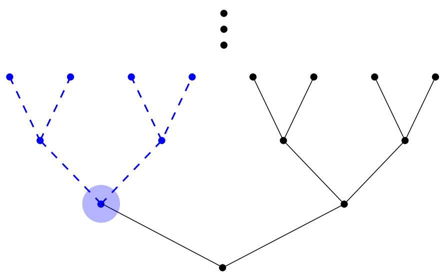

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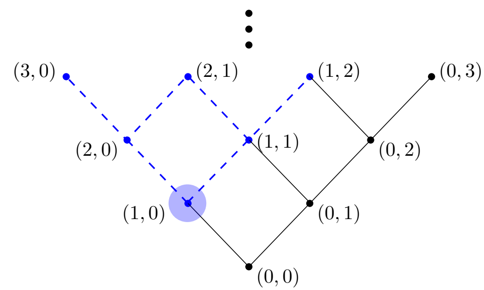

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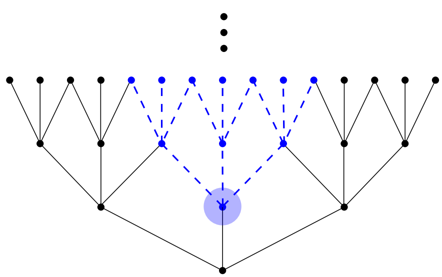

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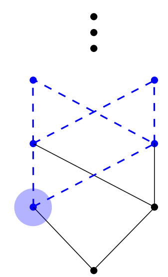

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Overview

We will discuss three topics as they relate to upho posets:

1. Schur-positivity of the Ehrenborg quasisymmetric function
2. Planar upho posets
3. Uncomputable rank-generating functions

The Ehrenborg Quasisymmetric Function

Definition

For any ranked poset P of finite type with a unique minimal element, define its **Ehrenborg quasisymmetric function** of degree n to be

$$E_{P,n}(x_1, x_2, \dots x_k) := \sum_{\substack{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_{k-1} < t_k \\ \rho(t_k)=n}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)}$$

where $\rho(t_i, t_{i+1}) = \rho(t_{i+1}) - \rho(t_i)$ and $\rho(t_i)$ is the rank of t_i. We also write $E_P := \sum_{n \geq 0} E_{P,n}$.

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Lemma

Let P be upho. Then E_P is a symmetric function. Moreover,

$$E_P(x_1, x_2, \ldots) = F_P(x_1) F_P(x_2) \cdots .$$

Criterion for Schur-Positivity

Theorem (Davydov, 2000)

An integral series $f(t) \in 1 + t\mathbb{Z}[[t]]$ *is totally positive, i.e.* $f(t_1)f(t_2)\cdots$ *is Schur positive, if and only if it is of the form* $f(t) = g(t)/h(t)$ *where* $g(t), h(t) \in \mathbb{Z}[t]$ *such that all the complex roots of* $g(t)$ *are negative real numbers and all the complex roots of* $h(t)$ *are positive real numbers.*

A Family of Schur-Positive Upho Posets

Theorem

Given positive integers $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_m$*, there exists an upho poset* P *with rank-generating function*

$$\frac{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}{(1-b_1x)(1-b_2x)\cdots(1-b_mx)}.$$

Proof of a Special Case

We will prove the following lemma:

Lemma

Given positive integers $a_1, a_2, \ldots, a_n$*, there exists an upho poset* Q *with rank-generating function*

$$F_Q(x) = \frac{(1 + a_1x)(1 + a_2x)\ldots(1 + a_nx)}{1 - x}.$$

Proof of a Special Case

- $S = \{(y_1, y_2, \ldots, y_n) \,|\, 0 \leq y_i \leq a_i \text{ for all } 1 \leq i \leq n\}$.

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- Rank 0 consists of $(0, \ldots, 0; 0)$, and $(y_1, y_2, \ldots, y_n; k) \lessdot (z_1, z_2, \ldots, z_n; k+1)$ if the two differ in at most one coordinate.

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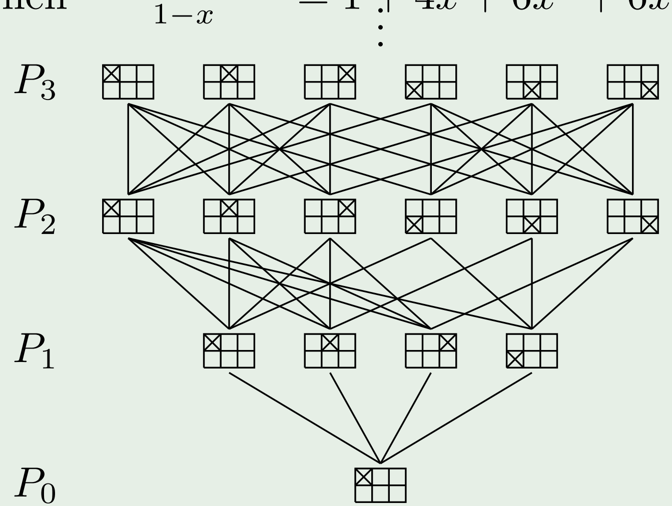
Example

Let $a_1 = 1$, $a_2 = 2$. Then $\frac{(1+x)(1+2x)}{1-x} = 1 + 4x + 6x^2 + 6x^3 + \cdots$.



Completing the Proof

We have constructed upho posets with rank-generating function

$$\frac{(1 + a_1 x)(1 + a_2 x) \cdots (1 + a_n x)}{1 - x}.$$

We want to extend this to

$$\frac{(1 + a_1 x)(1 + a_2 x) \cdots (1 + a_n x)}{(1 - b_1 x)(1 - b_2 x) \cdots (1 - b_m x)}.$$

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2. If P and Q are upho, $P \times Q$ is upho with $F_{P\times Q} = F_P F_Q$.
3. Multiply by c-trees with rank-generating function $1 + cx + c^2x^2 + \ldots = \frac{1}{1-cx}$.

□

Planarity

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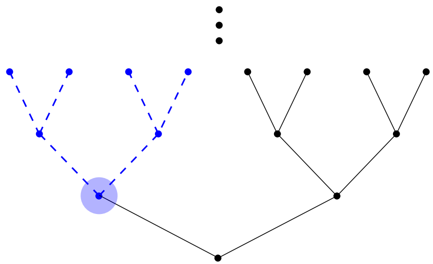

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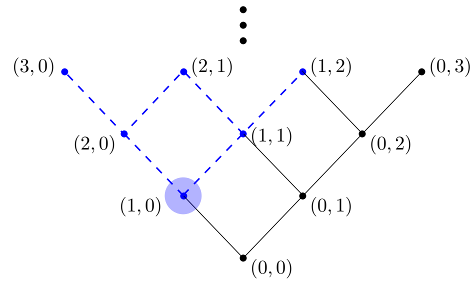

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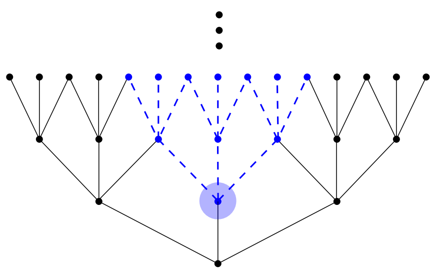

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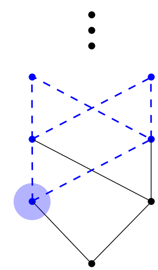


“Bowtie” Poset

Main Planarity Result

Theorem

The rank-generating function of any planar upho poset P with up-degree b is of the form

$$\frac{1}{Q(x)} = \frac{1}{1 - bx + a_2x^2 + a_3x^3 + \cdots + a_nx^n}$$

such that $b, a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ *and* $Q(1) \leq 0$. *Furthermore, any such function* $Q^{-1}(X)$ *is realized by some planar upho poset.*

Some Definitions

Definition

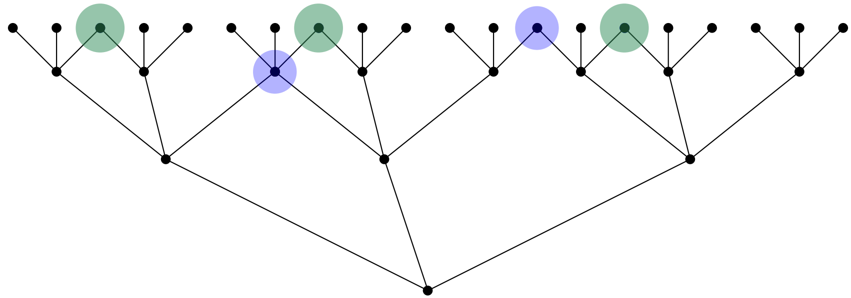
An element v of a planar upho poset is called **root-bifurcated** if it covers exactly 2 adjacent elements with greatest lower bound $\hat{0}$.

Definition

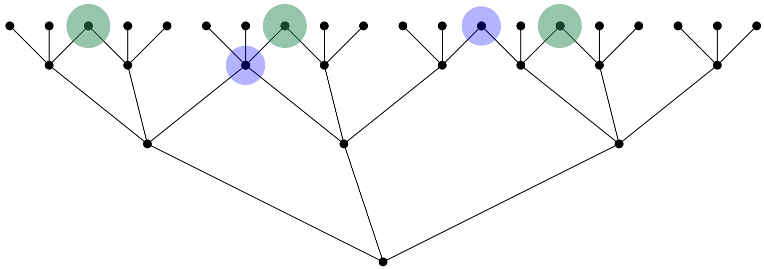
An element v of a planar upho poset is called **bifurcated** if it covers exactly 2 adjacent elements that do not have greatest lower bound $\hat{0}$.

Definition

An element v of a poset is an **atom** if it covers $\hat{0}$.



Proving the First Half of the Theorem



Proof.

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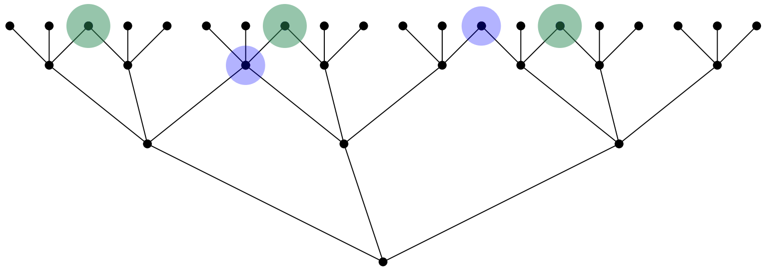

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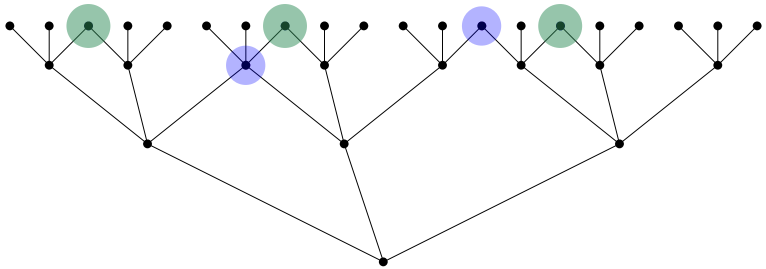

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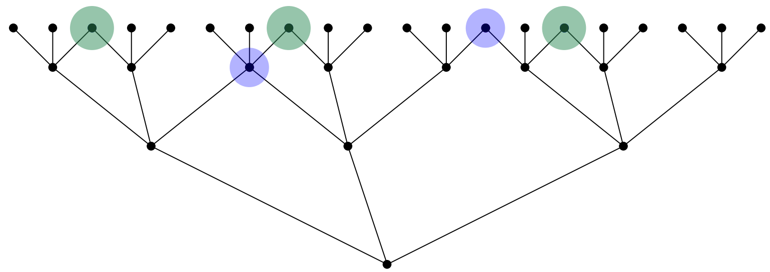

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2. There are $\leq b-1$ root-bifurcated elements in a poset with up-degree b.
3. If a_i is the number of root-bifurcated elements on rank i, then show that the rank-generating function of a planar poset P with up-degree b is

$$\left(1 - bx + \sum_{j=2}^{\infty} a_j x^j\right)^{-1}.$$

□

Uncomputability

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Theorem

There exists an upho poset with uncomputable rank-generating function.

Using Monoids to Construct Posets

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A **monoid** is a set that is closed under an associative binary operation and an identity element.

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- Alphabet $\Sigma = \{a_1, a_2, \ldots, a_b\}$, homogeneous relations $b_1 b_2 \ldots b_k = c_1 c_2 \ldots c_k$ for $b_i, c_i \in \Sigma$ for all $1 \leq i \leq k$.

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- Define a monoid M of finite strings of letters of Σ with the operation of concatenation.

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- $X = Y$ implies $AXB = AYB$, where A, B, X, and Y are strings of letters from Σ.
- Define a monoid M of finite strings of letters of Σ with the operation of concatenation.
- Define a poset P consisting of elements of M with the order relation $\leq$, where $X \leq Y$ for $X, Y \in M$ if $XA = Y$ for some $A \in M$.

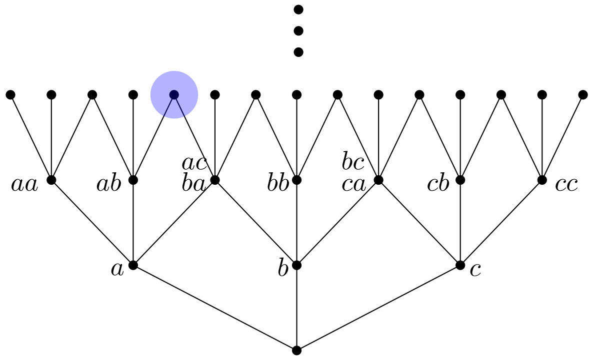
Examples

Example

Stern's poset is defined by the alphabet $\{a, b, c\}$ and the relations $ac = ba$ and $bc = ca$. Note, for example, that $baa = aca = abc$.

Example

A binary tree is defined by the alphabet $\{a, b\}$ and no relations.



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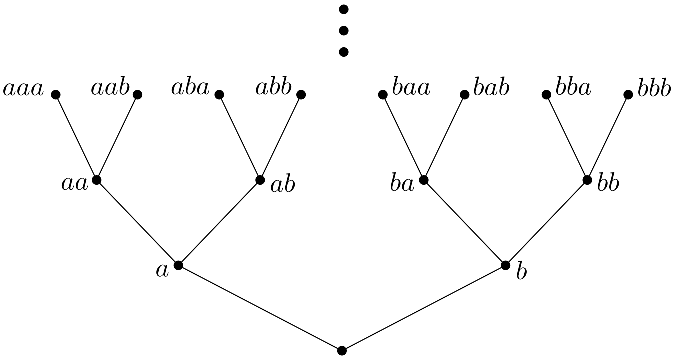

Binary tree

The Upho Condition in Relations

Lemma

Take a monoid M and its associated poset $P(M)$. Then, if $AX = AY$ implies $X = Y$ for all $A, X, Y \in M$, then $P(M)$ is upho.

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Lemma

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Proof.

1. $BX \leq BY$ if and only if $X \leq Y$, so the map $\iota : X \to BX$ is a homomorphism from P to V_B.
2. ι is also a bijection.
3. Thus, $V_B \cong P$ as desired.

□

Sketching the Proof

Proof.

1. Find an infinite set of relations S on an alphabet $\{L, R\}$:

$$LRLRLL = RRLLRL$$
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such that every poset defined by any subset of S is upho and has *distinct* rank-generating function.

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such that every poset defined by any subset of S is upho and has *distinct* rank-generating function.

2. This set of posets has an uncountably infinite number of different rank-generating functions.
3. There are a countably infinite number of computable rank generating functions (Sipser, 1996).

□

Acknowledgements

We would like to thank:

- Yibo Gao
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- The MIT PRIMES program