Lebesgue Measure Preserving Thompson's Monoid

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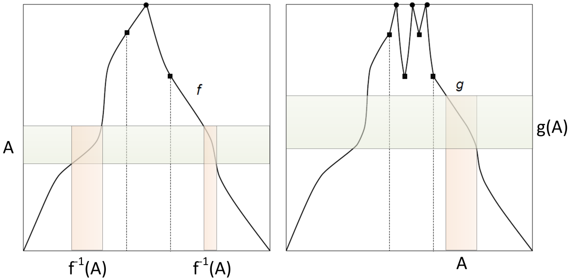
Lebesgue Measure Preserving Interval Maps

Continuous $h \colon [0,1] \xrightarrow{\text{onto}} [0,1]$. λ-preserving if $\forall A \in \mathcal{B}, \lambda(A) = \lambda(h^{-1}(A))$.

- λ: Lebesgue measure on $[0,1]$. $\mathcal{B}$: Borel sets on $[0,1]$.



- The above definition does not imply $\lambda(A) = \lambda(h(A))$. In fact, if h is λ-preserving, $\lambda(A) \leq \lambda(h(A))$ for any $A \in \mathcal{B}$.

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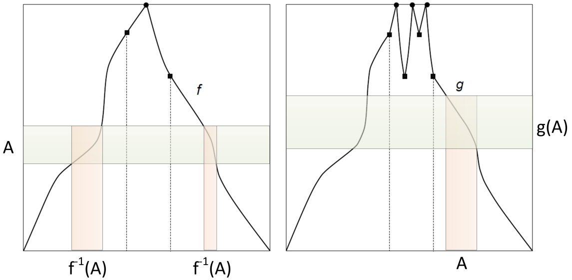

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Dynamical System

Topological dynamical system: $h^n = \underbrace{h \circ h \circ \cdots \circ h}_{n \text{ times}}$.

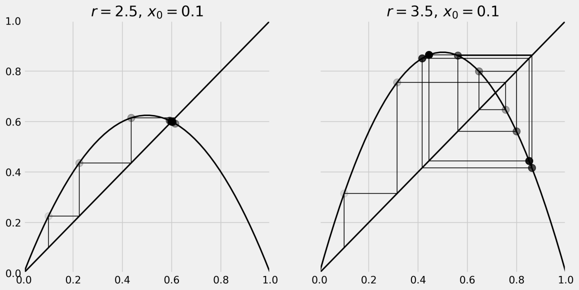
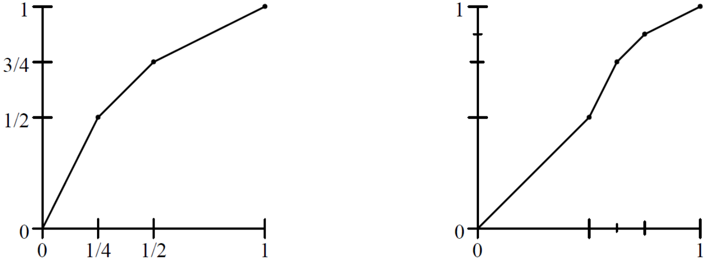


Figure: Logistic map $x_{n+1} = rx_n(1 - x_n)$, which is NOT λ-preserving.

Thompson's Group $\mathbb{F}$

Continuous function $f\colon [0,1] \xrightarrow{\text{onto}} [0,1]$. Piecewise affine, dyadic breakpoints, derivative $= 2^k$ with integer k.



Any $f \in \mathbb{F}$ is generated by the above two generator maps.

λ-Preserving Thompson's Monoid $\mathbb{G}$

- $\mathbb{F}$ maps and λ-preserving maps do not naturally intersect.
  - Except for the identity map, any $\mathbb{F}$ map does not preserve λ and any λ-preserving map does not preserve orientation and thus does not belong to $\mathbb{F}$.
- We propose λ-preserving Thompson's monoid, $\mathbb{G}$, which is similar to $\mathbb{F}$ except that the derivatives of piecewise affine maps can be negative to preserve λ, i.e., $\pm 2^k$ for integer k.
  - Monoids are semigroups with a single associative binary operation and an identity element.
  - Unlike $\mathbb{F}$, $\mathbb{G}$ maps are non-invertible except for trivial maps.
- Monoid $\mathbb{G}$ has not been proposed or studied in the literature and exhibits very different *algebraic and dynamical properties* from $\mathbb{F}$ or λ-preserving interval maps in general.

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Properties of Monoid $\mathbb{G}$

In this project we have studied the following properties

- Algebraic properties
  - Approximation
  - Entropy
  - Decomposition, equivalence classes and finitely generated monoid
- Dynamical properties
  - Mixing
  - Periodic points
  - Topological conjugacy

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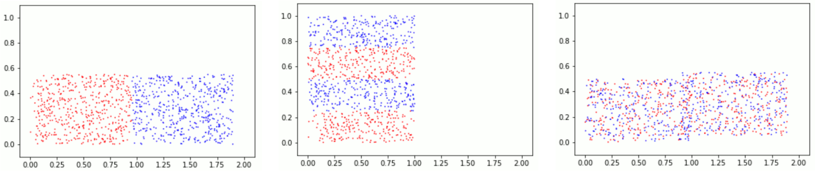
Mixing — Illustration

Mixing process: an example.



Repeated application of the baker's map to points colored red and blue, initially separated. After several iterations, the red and blue points seem to be completely mixed.

Mixing — Theorems (1)

Definition (*Topological Mixing (TM)*)

An interval map h is TM if for all nonempty open sets U, V in $[0,1]$, $\exists N \geq 0$ such that $\forall n \geq N$, $f^n(U) \cap V \neq \emptyset$.

Definition (*Locally Eventually Onto (LEO)*)

An interval map h is LEO if for every nonempty open set U in $[0,1]$ there is an integer N such that $h^N(U) = [0,1]$.

In general, LEO implies TM and the converse does not hold. However, we prove that the two are equivalent for $g \in \mathbb{G}$:

Theorem

If $g \in \mathbb{G}$ is TM, then g is LEO.

Mixing — Theorems (2)

Definition

$C(\lambda)$: set of continuous λ-preserving maps.

Definition

$\rho(h_1, h_2) = \sup_{x \in [0,1]} |h_1(x) - h_2(x)|$. If $\rho(h_1, h_2) < \epsilon$, h_2 is said to be within ϵ neighborhood of h_1.

Theorem

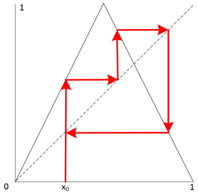
Denote by $\mathbb{G}_{LEO}$ the subset of $\mathbb{G}$ whose elements are LEO. $\mathbb{G}_{LEO}$ is dense in $C(\lambda)$.

The theorem states that $\forall h \in C(\lambda)$ and $\epsilon > 0$, there exists $g \in \mathbb{G}_{\text{LEO}}$ such that $\rho(h, g) < \epsilon$.

Periodic Points — Definition and Theorem

Definition (*Preperiodic and Periodic Points*)

Point x is *preperiodic* if $\exists n > m > 0$ such that $h^n(x) = h^m(x)$. If $m = 0$, then x is *periodic*.



Theorem

On any $g \in \mathbb{G}$, if c is dyadic, then point $(c, g(c))$ is preperiodic.

Markov Maps — Definition and Theorem

Definition (*Markov Map*)

A piecewise affine map is Markov if all breakpoints are preperiodic.

Theorem

Any $g \in \mathbb{G}$ is a Markov map.

Period 3 Implies Chaos & Characterization of $\mathbb{G}$ Maps

Definition (*Period of a Periodic Point*)

The period of periodic point x is the least positive integer p such that $h^p(x) = x$.

Definition (*Chaotic Function*)

Map h is chaotic if for any $k > 0$, point x of period k exists.

- Li-Yorke theorem (1975) states that if a periodic point x of period 3 exists, then h is chaotic.

We characterize periods of periodic points of all maps in $\mathbb{G}$

- Maps in one specific subset of $\mathbb{G}$ always have periodic points with period 3.
- For any remaining map, $\exists$ odd n_0 such that there exist any odd period $n \geq n_0$, period $n = 1$ and any even period n.

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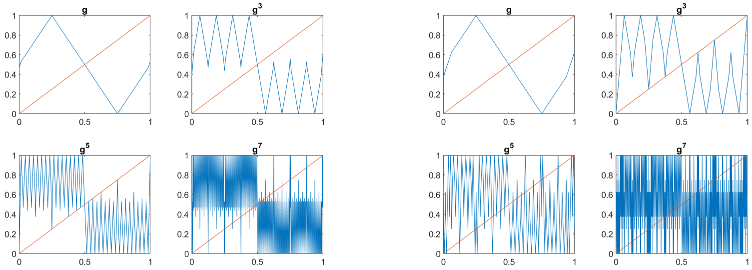
Examples of Periodicity of $\mathbb{G}$ Maps

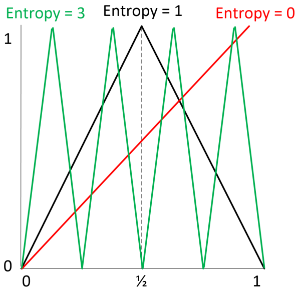


- (Left) Periodic points of period 3 do not exist but periodic points of periods 5 and 7 exist.
- (Right) Periodic points of period $3, 5, 7$ all exist.

Entropy — Definition

Definition (*Entropy*)

$$c_\lambda(h) = \int_0^1 \log_2 |h'(x)| \, d\lambda(x)$$



Entropy — Prior Result

Definition

PA(λ): set of piecewise affine λ-preserving maps.

Bobok and Troubetzkoy (2019) showed that $\forall c \in (0, \infty)$, Markov LEO *PA*$(\lambda)$ is dense in $C(\lambda)$ with $c_\lambda(h) = c$.

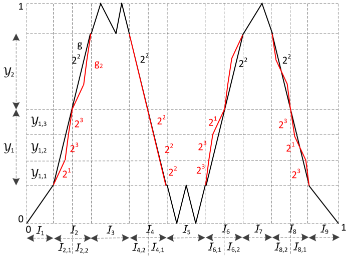
What entropy range can $\mathbb{G}$ *achieve?*

Entropy — Minimization

Suppose that g on $g^{-1}(\mathcal{Y})$ is m affine legs with absolute values of the derivatives equal to $\{2^{k_i}\}$. To minimize $c_\lambda(g)$ with $g \in \mathbb{G}$,

$$\left\{\begin{array}{l} \min_{k_1,\ldots,k_m} \sum_{i=1}^{m} k_i 2^{-k_i}, \\ \text{s.t. } \sum_{i=1}^{m} 2^{-k_i} = 1 \end{array}\right. \Rightarrow k_i^* = \left\{\begin{array}{l} i, i = 1, 2, \ldots, m-1 \\ m-1, i = m. \end{array}\right.$$

Key idea: Any set of m affine legs of $\{2^{k_i}\}$ can be replaced by another set of $\{2^{k_i^*}\}$ within ϵ neighborhood; converse is not true.



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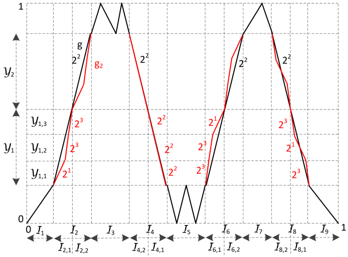

Entropy — Theorem

Theorem

For any $c \in [2, \infty)$ and $\epsilon > 0$, the set of Markov LEO maps in $\mathbb{G}$ whose entropy is within ϵ of c is dense in $C(\lambda)$.

- With $\{2^{k_i^*}\}$, minimum $c_\lambda(g)$ is given by $\sum_{i=1}^{m-1} i2^{-i} + (m-1)2^{-(m-1)} < 2$ for any m.
- Maximum $c_\lambda(g)$ is unbounded.

Compared with $c_\lambda(h)$, the constraints on $\mathbb{G}$ lead to

- $c_\lambda(g)$ can only be within ϵ of, but may not be exactly equal to, target c.
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Acknowledgments

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