Homomorphisms of Graphs: Colorings, Cliques and Transitivity

Minghan Sun, Andrew Weinfeld, Christopher Zhu

MIT PRIMES
Reading Group Mentor: Younhun Kim

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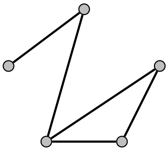
Graphs and Homomorphisms

What is a graph?

A **graph** X is a collection of **vertices** (dots) and **edges** (line segments or arrows).



Notation:

- $V(X)$: the set of vertices.
- $E(X)$: the set of edges.
- $u \sim v$: edge $\{u, v\} \in E(X)$.

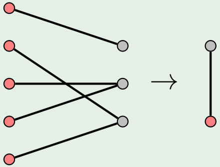
Graph Homomorphisms

Definition

Let X and Y be graphs. A map $\varphi : V(X) \to V(Y)$ is a homomorphism if $\varphi(x) \sim \varphi(y)$ whenever $x \sim y$. Less formally, a homomorphism maps edges to edges.

Example

φ :



Colorings

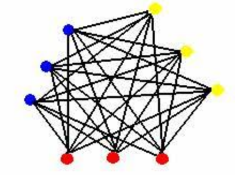
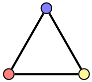
Definition

Let I be an subset of the vertex set $V(G)$ of a graph. We say that I is an independent set if there exists no edge that joins two vertices in I.

Definition

For a positive integer c, a c-coloring of a graph G is a partition of $V(G)$ into c independent sets. The chromatic number of a graph, $\chi(G)$, is the smallest integer n such that G has a n-coloring.

We can think of a c-coloring of G as a homomorphism $G \to K_c$ that identifies each independent set with a distinct vertex of K_c.

φ :  $\to$ 

Hedetniemi's Conjecture

- $\psi : X \to Y$ exists, $\implies \chi(X) \le \chi(Y)$, because there is $\pi : Y \to K_{\chi(Y)}$, and $\pi \circ \psi$ is a homomorphism $X \to K_{\chi(Y)}$.
- Since the map that sends (x, y) to x is a homomorphism $X \times Y \to X \implies \chi(X \times Y) \le \min\{\chi(X), \chi(Y)\}$.

Conjecture (Hedetniemi, 1966)

For all graphs X, Y, we have $\chi(X \times Y) = \min\{\chi(X), \chi(Y)\}$.

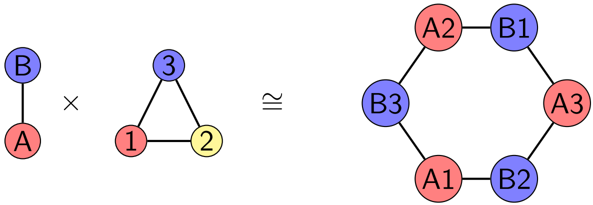


Figure: $K_2 \times K_3 \cong C_6$

Shitov's counterexample (2019)

Main Idea:

1. Shitov proves that if G contains a large cycle but no short ones,

$$\chi(\varepsilon_c(G \boxtimes K_q)) > c \tag{1}$$

where $c = \lceil 3.1 \cdot q \rceil$.

2. Can also show:

$$\chi(G \boxtimes K_q) > c \tag{2}$$

3. and yet...

$$\chi((G \boxtimes K_q) \times \varepsilon_c(G \boxtimes K_q)) = c. \tag{3}$$

Future Directions

There have been attempts to modify Hedetniemi's Conjecture, in terms of the *Poljak–Rödl function*.

Definition

The Poljak–Rödl function $f : \mathbb{N} \to \mathbb{N}$ satisfies

$$f(n) = \min_{\chi(G),\chi(H) \geq n} \chi(G \times H). \tag{4}$$

Hedetniemi is false $\implies f(n) < n$ for some $n \in \mathbb{N}$.

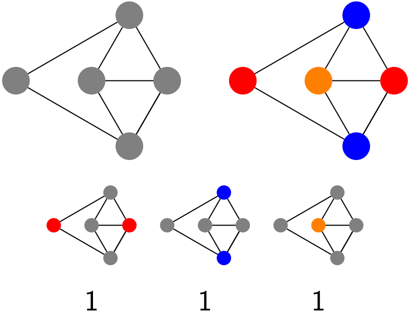
Weak Hedetniemi Conjecture

$$\lim_{n \to \infty} f(n) = \infty. \tag{5}$$

Colorings and Cliques

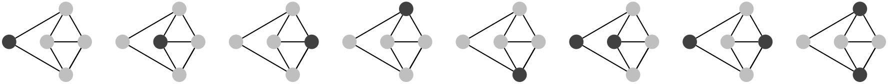
Generalizing Colorings

Standard k-coloring of a graph = a $\{0, 1\}$-valued function on independent sets

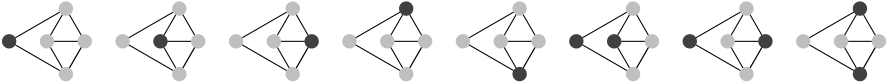


Generalization: a nonnegative function on *all* independent sets of a graph.

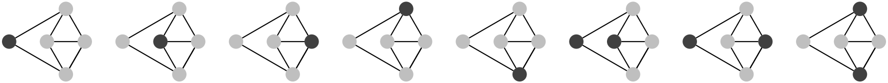
Fractional Colorings: Examples



$$\left(0,\quad 1,\quad 0,\quad 0,\quad 0,\quad 0,\quad 1,\quad 1\right)$$



$$\left(1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1\right)$$



$$\left(0,\quad 0.5,\quad 0.5,\quad 0,\quad 0,\quad 0.5,\quad 0.5,\quad 1\right)$$

Fractional Colorings: Definitions

Let $\mathcal{I}(X)$ denote the set of all independent sets of a graph X, and let $\mathcal{I}(X, u)$ denote all the independent sets that also contain the vertex u.

Definition

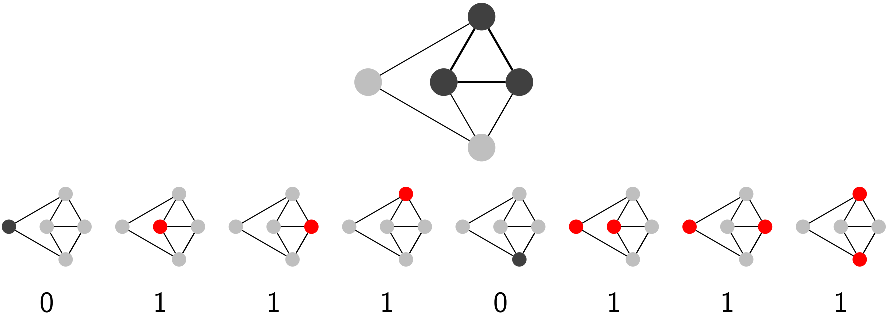
A fractional coloring of a graph X is a function $f : \mathcal{I}(X) \to \mathbb{R}_{\geq 0}$ such that for all vertices $x \in X$, $\sum_{S \in \mathcal{I}(X,x)} f(S) \geq 1$.

Definition

The weight of a fractional coloring is defined as $\sum_{S \in \mathcal{I}(X)} f(S)$. The fractional chromatic number $\chi^*(X)$ of the graph X is the minimum possible weight of a fractional coloring.

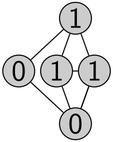
Generalizing Cliques

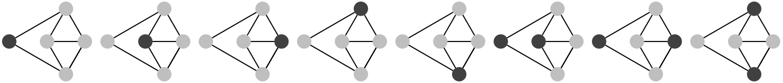
Cliques (complete subgraphs) = $\{0, 1\}$-valued functions on vertices.



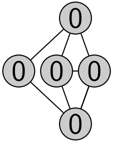
Generalization: sum up *nonnegative* functions over vertices.

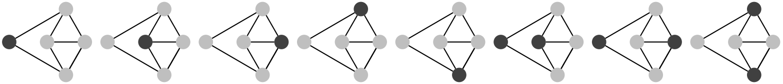
Fractional Cliques: Examples



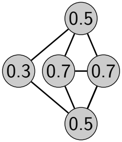


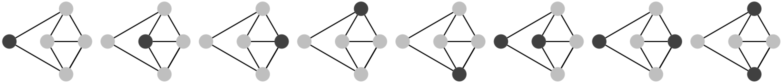
$$\left(0,\quad 1,\quad 1,\quad 1,\quad 0,\quad 1,\quad 1,\quad 1\right)$$





$$\left(0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0,\quad 0\right)$$





$$\left(0.3,\quad 0.7,\quad 0.7,\quad 0.5,\quad 0.5,\quad 1,\quad 1,\quad 1\right)$$

Fractional Cliques: Definitions

Definition

A fractional clique of a graph X is a function $f : V(X) \to \mathbb{R}_{\geq 0}$ such that $\sum_{v \in V(S)} f(v) \leq 1$ for all independent sets $S \in \mathcal{I}(X)$.

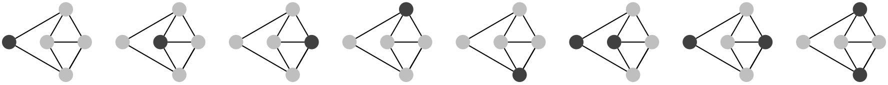
Definition

The weight of a fractional clique is defined as $\sum_{v \in V(X)} f(v)$. The fractional clique number of $\omega^*(X)$ of the graph X is the maximum possible weight of a fractional clique.

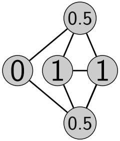
Duality

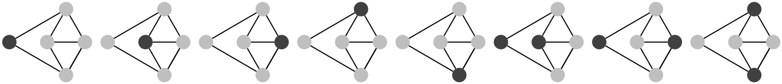
Proposition

For any graph X, we have $\omega^*(X) \le \chi^*(X)$.



$$\big(0,\quad 0.5,\quad 0.5,\quad 0,\quad 0,\quad 0.5,\quad 0.5,\quad 1\big)$$





$$\big(0,\quad 1,\quad 1,\quad 0.5,\quad 0.5,\quad 1,\quad 1,\quad 1\big)$$

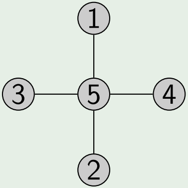
Symmetry of graphs: Transitivity

Graph Automorphisms

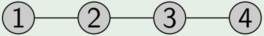
Definition

A graph automorphism is a permutation of the vertices that takes edges to edges and nonedges to nonedges. They form a group, $Aut(X)$.

Example



$Aut(X) = S_4$



$Aut(X) = \{(1), (14)(23)\}$

Proposition

A graph automorphism preserves the degree of a vertex.

Transitivity

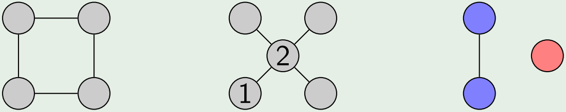
$Aut(X)$ acts on the set of vertices, the set of edges, and the set of arcs (ordered pairs of two adjacent vertices).

Definition

Given a set A on which $Aut(X)$ acts, we say that a graph is A-transitive if for every $a, b \in A$, there is a graph automorphism taking a to b.

Example

Any cycle C_n is vertex, edge, and arc transitive.
The star graph $K_{1,4}$ is edge but not arc transitive since $(1,2) \not\to (2,1)$.
The graph $C_2 \cup C_1$ is arc and edge transitive but not vertex transitive.



C_4 $K_{1,4}$ $C_2 \cup C_1$

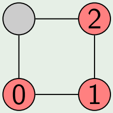
Definition

An *s-arc* is a sequence $(v_0, v_1, \ldots, v_s)$ of adjacent vertices such that $v_{i-1} \neq v_{i+1}$ for all i.

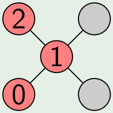
Note that 0-arc transitivity is the same as vertex transitivity, and 1-arc transitivity is the same as arc transitivity.

Example

A cycle C_n, $n \geq 3$ is s-arc transitive for all s.
The star graph $K_{1,4}$ is 2-arc transitive.



C_4

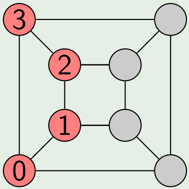


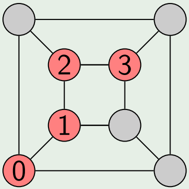
$K_{1,4}$

s-arc Transitive Graphs

Example

The cube is 0-, 1-, and 2-arc transitive, but not 3-arc transitive.





Proposition

If every connected component of X contains a cycle, then

$$s\text{-arc transitive} \implies (s-1)\text{-arc transitive.}$$

If X satisfies this condition and is s-transitive for some s, then X is vertex transitive, so every vertex has the same degree.

We will consider graphs of degree at least 3.

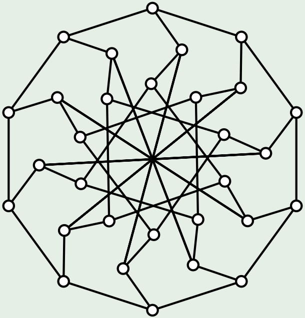
Restrictions on s

Theorem (Tutte, 1947)

Let X be an s-arc transitive graph of degree equal to 3. Then $s \leq 5$.

Example

The Tutte-Coxeter graph achieves $s = 5$.



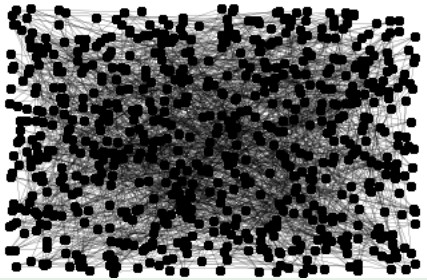
Restrictions on s

Theorem (Weiss, 1981)

Let X be an s-arc transitive graph of degree at least 3. Then $s \leq 7$. Furthermore, if $s = 6$ then X is 7-arc transitive.

Example

The smallest known example of a nontrivial 7-arc transitive graph has degree four and is on 728 vertices.



Thank you

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