Induced Representations of Finite Groups

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MIT PRIMES Conference: December 2020

Linear Representations

Definition

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Example

- Let $C_n = \{g^m \mid 0 \le m < n\}$ be the cyclic group.
  $\rho : C_n \to \mathbb{C}^\times$, $\rho(g^k) = e^{2\pi i \frac{k}{n}}$, $0 \le k < n$, for every $g \in G$.

G-invariant Subspaces

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation over $\mathbb{C}$.

Definition (G-invariant subspace)

A linear subspace W of V is called G-invariant if $\rho(g)(W) \subseteq W$ for all $g \in G$.

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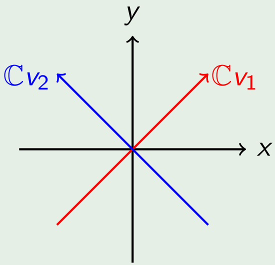
Example

$$\rho : C_2 \to \mathrm{GL}_2(\mathbb{C}), \gamma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Eigenvectors to $+1$ and -1:
$v_1 = (1, 1)$, $v_2 = (-1, 1)$.



Definitions and Maschke's Theorem

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Theorem (Maschke)

Every complex linear representation is the **direct sum** *of irreducible representations.*

Character Theory

Definition (Character of a representation)

The **character** of a linear representation $\rho : G \to GL(V)$ is the complex valued function $\chi : G \to \mathbb{C}$, given by

$$\chi_\rho(s) := \mathrm{Tr}(\rho(s))$$

for every $s \in G$.

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- The character is a class function on G.
- The space $\mathbf{H}$ of class functions on G has a scalar product given by $\langle f, f' \rangle := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{f'(g)}$, for $f, f' \in \mathbf{H}$.

Character Theory

Theorem (Orthogonality of Characters)

Let χ_ρ and $\chi_{\rho'}$ be the characters of the irreducible representations ρ and ρ', respectively. Then, $\langle \chi_\rho, \chi_{\rho'} \rangle = 1$ if ρ and ρ' are equivalent and $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$ if they are not.

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- $\langle \chi_W, \chi_V \rangle = \dim \operatorname{Hom}_G(W, V)$ for a $\mathbb{C}G$-module V and a simple $\mathbb{C}G$-module W.

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- Two representations ρ and ρ' are equivalent iff $\chi_\rho = \chi_{\rho'}$.
- The characters of all irreducible representations of G form an orthonormal basis of $\mathbf{H}$.
- The number of irreducible representations of G is equal to the number of conjugacy classes of G.

Induced Representations - Definition

Let $\theta : H \to \mathrm{GL}(W)$ be a representation. Select a **system of representatives** $R := \{\sigma \in gH : gH \in G/H\}$ of G/H and set $W_\sigma := \mathbb{C}\sigma \otimes_{\mathbb{C}} W$. Construct a new representation

$$\tau : G \to \mathrm{GL}(\bigoplus_{\sigma \in R} W_\sigma)$$

by

$$\tau(t)(\sigma \otimes w) = t\sigma \otimes w = \sigma' \otimes \theta(h)w$$

where $t\sigma = \sigma' h$ with $\sigma' \in R$ and $h \in H$.

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Definition

A representation $\rho : G \to \mathrm{GL}(V)$ is **induced** by $\theta : H \to \mathrm{GL}(W)$ if $V \cong \bigoplus_{\sigma \in R} W_\sigma$ as representations of G.

Induced Representations - Alternative Definition

Definition

Let $\theta : H \to \mathrm{GL}(W)$ be a linear representation which equips W with the structure of a left $\mathbb{C}H$-module. Set $V = \mathbb{C}G \otimes_{\mathbb{C}H} W$. The representation $\mathrm{Ind}_H^G(\theta) : G \to \mathrm{GL}(V)$ given by

$$\mathrm{Ind}_H^G(\theta)(g)(\sigma \otimes w) = g\sigma \otimes w$$

is called **induced** by θ.

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- As $\bigoplus_{\sigma \in R} W_\sigma \cong \mathbb{C}G \otimes_{\mathbb{C}H} W$, both definitions are equivalent.
- If f is a class function on H, the function defined by $\mathrm{Ind}_H^G(f)(u) := \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}ut \in H}} f(t^{-1}ut)$ for every $u \in G$ is the **induced class function** on G.

Examples of Induced Representations

Example

The regular representation r_G of G is induced by the regular representation r_H of every $H \subset G$: $\mathbb{C}G \cong \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}H$.

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Example

Let $G = S_3$, $H = \mathbb{Z}_2$. Let τ be the signum rep. of H, let ϵ be the signum rep. of G and let ρ be the standard rep. of G. Then $\mathrm{Ind}_H^G(\tau) = \epsilon \oplus \rho$.

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Example

For representations $\theta_i : H \to \mathrm{GL}(W_i)$, $i = 1, 2$, of H, $\mathrm{Ind}_H^G(\theta_1 \oplus \theta_2) = \mathrm{Ind}_H^G(\theta_1) \oplus \mathrm{Ind}_H^G(\theta_2)$.

Characters of Induced Representations

Theorem

Let $\theta : H \to \mathrm{GL}(W)$ *be a representation of* $H \subset G$ *and* R *a system of representatives of* G/H*. Then, for each* $u \in G$*, we have*

$$\mathrm{Ind}_H^G(\chi_\theta)(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \chi_{\mathrm{Ind}_H^G(\theta)}(u).$$

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Example

- $\mathrm{Ind}_H^G(\chi_{r_H}) = \chi_{r_G}$.
- $\mathrm{Ind}_H^G(\chi_{\theta_1 \oplus \theta_2}) = \mathrm{Ind}_H^G(\chi_{\theta_1}) \oplus \mathrm{Ind}_H^G(\chi_{\theta_2}) = \chi_{\mathrm{Ind}_H^G(\theta_1)} \oplus \chi_{\mathrm{Ind}_H^G(\theta_2)}$.

Frobenius Reciprocity

Theorem (Frobenius Reciprocity)

Let E and W be a $\mathbb{C}G$-module and a $\mathbb{C}H$-module, respectively. Then, we have

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G W, E) \cong \mathrm{Hom}_H(W, \mathrm{Res}_H^G E).$$

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Corollary (Frobenius Reciprocity for Characters)

$\langle \mathrm{Ind}_H^G \chi_\rho, \chi_{\rho'} \rangle_G = \langle \chi_\rho, \mathrm{Res}_H^G \chi_{\rho'} \rangle_H.$

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Corollary (Frobenius Reciprocity for Characters)

$\langle \mathrm{Ind}_H^G \chi_\rho, \chi_{\rho'} \rangle_G = \langle \chi_\rho, \mathrm{Res}_H^G \chi_{\rho'} \rangle_H.$

Frobenius Reciprocity states that if ρ and ρ' are irreducible representations of H and G, respectively, then the multiplicity of ρ' in $\mathrm{Ind}_H^G(\rho)$ equals the multiplicity of ρ in $\mathrm{Res}_H^G(\rho')$.

Example (2-dimensional irreducible representation of D_4)

$$\rho : r^k \mapsto \begin{pmatrix} e^{2\pi ik/4} & 0 \\ 0 & e^{-2\pi ik/4} \end{pmatrix}$$

$$sr^k \mapsto \begin{pmatrix} 0 & e^{-2\pi ik/4} \\ e^{2\pi ik/4} & 0 \end{pmatrix} \quad \text{for all } k = 0, 1, 2, 3.$$

The cyclic subgroup $C_4 \leq D_4$ has an irreducible representation $\rho_1 : C_4 \to \mathbb{C}^\times$ with character $\chi_{\rho_1}(r^k) = e^{2\pi ik/4}$ for $k = 0, 1, 2, 3$. By Frobenius reciprocity,

$$\begin{aligned} \langle \mathrm{Ind}_{C_4}^{D_4}(\chi_{\rho_1}), \chi_\rho \rangle &= \langle \chi_{\rho_1}, \mathrm{Res}_{C_4}^{D_4}(\chi_\rho) \rangle \\ &= \frac{1}{4}(2 + 1 + e^{\pi i} + 1 + e^{2\pi i} + 1 + e^{3\pi i}) = 1. \end{aligned}$$

Hence, the irreducible ρ of D_4 is induced by the irreducible ρ_1 of C_4.

Counterexample

Question: Is the induced representation of an irreducible representation always irreducible?

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Answer: No!

Example

Let $G = S_3$, $H = \mathbb{Z}_2$. Let τ be the signum representation of H, let ϵ be the signum representation of G and let ρ be the standard representation of G. We can compute

$$\begin{aligned} \chi_{\mathrm{Ind}_H^G(\tau)}(\mathrm{Id}) &= 3, & \chi_{\mathrm{Ind}_H^G(\tau)}((12)) &= -1, & \chi_{\mathrm{Ind}_H^G(\tau)}((123)) &= 0. \\ \chi_\epsilon(\mathrm{Id}) &= 1, & \chi_\epsilon((12)) &= -1, & \chi_\epsilon((123)) &= 1. \\ \chi_\rho(\mathrm{Id}) &= 2, & \chi_\rho((12)) &= 0, & \chi_\rho((123)) &= -1. \end{aligned}$$

$$\chi_{\mathrm{Ind}_H^G(\tau)} = \chi_\epsilon + \chi_\rho.$$

Mackey's Irreducibility Criterion

Let $\rho : H \to GL(W)$, $H \leq G$, be a representation.
Let $H_s := sHs^{-1} \cap H$ for $s \in G$.
Let $\rho^s : H_s \to \mathrm{GL}(W)$ be a representation given by $\rho^s(x) := \rho(s^{-1}xs)$ for $x \in H_s$.
Let $\mathrm{Res}_s(\rho)$ denote the restriction of ρ to H_s.

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Theorem (Mackey's Irreducibility Criterion)

In order that $\mathrm{Ind}_H^G(\rho)$ is an irreducible representation, it is **necessary and sufficient** *that the following two conditions be satisfied:*
(i) W is a simple left $\mathbb{C}H$-module.
(ii) For every $s \in G - H$, we have $\langle \rho^s, \mathrm{Res}_s(\rho) \rangle = 0$.

Acknowledgements

We would like to thank:

- Our mentor, Dr. Alexander Vitanov of MIT, for his valuable guidance throughout this year.
- MIT PRIMES for this wonderful opportunity.
- Prof. Pavel Etingof, Dr. Tanya Khovanova, and Dr. Slava Gerovitch for their advice and for organizing the PRIMES program.
- The MIT Math Department.

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