Alpha invariants of K-semistable smooth toric Fano varieties

Alvin Chen* Kai Huang †

Abstract

Jiang conjectured that the α-invariant for n-dimensional K-semistable smooth Fano varieties has a gap between $\frac{1}{n}$ and $\frac{1}{n+1}$, where $\frac{1}{n+1}$ can only be achieved by projective n-space. Assuming a weaker version of Ewald's conjecture, we prove this gap conjecture in the toric case. We also prove a necessary and sufficient classification for all possible values of the α-invariant for K-semistable smooth toric Fano varieties by providing an explicit construction of the polytopes that can achieve these values. This provides an important step towards understanding the types of polytopes that correspond to particular values of the α-invariant; in particular, we show that K-semistable smooth Fano polytopes are centrally symmetric if and only if they have an α-invariant of $\frac{1}{2}$. Lastly, we examine the effects of the Picard number on the α-invariant, classifying the K-semistable smooth toric Fano varieties with Picard number 1 or 2 and their α-invariants.

* North Carolina School of Science and Mathematics, Durham, NC 27705, USA
† Massachusetts Institute of Technology, Cambridge, MA 02139, USA

1 Introduction

One of the most important questions in differential geometry asks which manifolds admit a Kähler-Einstein metric. In the case where a manifold does admit a Kähler metric, there are three different cases to be considered depending on the first Chern class, an important classification related to complex vector bundles. When the first Chern class is negative, the manifold is considered *general type*, and it was proved by Aubin [2] and Yau [1] that general type Kähler manifolds all have a Kähler-Einstein metric. When the first Chern class is zero, these manifolds are called *Calabi-Yau*, and Yau [1] resolved this case and proved that all Calabi-Yau Kähler manifolds also all have Kähler-Einstein metrics. This is related to the Calabi conjecture, and Yau won the Fields medal in part for this work. The last case, when the first Chern class is positive, is the most difficult to deal with. These manifolds are called *Fano*.

It is not true that all Fano manifolds with a Kähler metric also admit a Kähler-Einstein metric. In fact, in the Fano case, the Kähler-Einstein metric was proved to be equivalent to a condition in algebraic geometry, called K-stability. This was a result of the Yau-Tian-Donaldson conjecture, which was recently resolved by Chen, Donaldson, and Sun [10].

As such, questions regarding K-stability were tied with geometric invariant theory, and work with stability conditions could be used to calculate whether a variety admits a Kähler-Einstein metric.

To better understand the manifolds and evaluate whether varieties admit the Kähler-Einstein metric, it is important to manipulate various necessary or sufficient conditions on varieties that allow determination of the K-stability of a variety. Tian [9] introduced the α-invariant, which gives a sufficient numerical condition on whether a variety has a Kähler-Einstein metric.

In some sense, the α-invariant measures the worst singularities that can lie on a variety, where singularities are points are abrupt points that disrupt the "smoothness" of a variety. Compared to other invariants related to K-stability, the α-invariant is relatively easy to calculate, and it translates nicely to a combinatorial condition when considering toric varieties.

In particular, this paper focuses on K-semistable smooth toric Fano varieties, a type of variety which gives way to a combinatorial formulation of the α-invariant. Applying K-semistability to the formulation is a novel technique that has enabled us to deduce important information on the behavior of the α-invariant.

The α-invariant is inherently tied to K-semistability. Tian [9] showed that X admits a Kähler-Einstein metric if $\alpha(X) > \frac{n}{n+1}$. Fujita [7] provided the equality case, showing that if $\alpha(X) = \frac{n}{n+1}$, then X is K-stable and thus admits Kähler-Einstein metrics.

When considering smoothness with regard to K-semistability, other inequalities can also be established relating the α-invariant to K-semistability. In fact, Jiang [8] proved that for small α-invariants X had to be a projective variety.

Theorem 1.1 ([8])**.** *If X is a K-semistable smooth Fano variety and $\alpha(X) \leq \frac{1}{n+1}$, then $X \cong \mathbb{P}^n$.*

Since $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$, this means that $\frac{1}{n+1}$ is the minimum possible value of the α-invariant for smooth K-semistable varieties.

Furthermore, Jiang made the following conjecture.

Conjecture 1.2 ([8])**.** *If X is a smooth K-semistable Fano variety and $\alpha(X) < \frac{1}{n}$, then $X \cong \mathbb{P}^n$.*

This conjecture would imply that there is a gap between $\frac{1}{n+1}$ and $\frac{1}{n}$ for the α-invariant on smooth K-semistable Fano varieties.

Our focus is on *toric* varieties, or varieties that contain a torus as an open dense subset and have the torus action act on the variety. Toric varieties are a crucial type of algebraic variety because they often serve as a testing ground and an important case of theorems in algebraic geometry. We use the polytopes of toric varieties to investigate the behavior of the α-invariant. Many common examples of varieties are toric, including affine space, projective space, and the products of projective space. we present the background of toric varieties and polytopes in Section 2.

In this paper, we use combinatorial methods to show that Jiang's conjecture is true for toric varieties, assuming a weaker form of the following conjecture (Ewald's conjecture).

Conjecture 1.3 ([6])**.** *Up to a unimodular transformation, all vertices of a Fano polytope have coordinates in $\{-1, 0, 1\}$. The polytope also contains all of the standard basis vectors e_i.*

A unimodular transformation is a transformation by a matrix with determinant ± 1. This is needed because it preserves the volume of the polytope. Essentially, this means that all vertices of a Fano polytope can be transformed inside a cube with side length 2, where one specific face maps to the standard basis.

In fact, we only need all vertices to have coordinates no less than -1, a weaker version of Ewald's conjecture. We describe the relationship between toric varieties and polytopes in Section 2.

Using this weakened conjecture and combinatorial inequalities, we prove Jiang's conjecture for the toric case.

Theorem 1.4. *Let P be a K-semistable smooth Fano polytope. Assume Conjecture 1.3 is true. Then, if X_P, the variety corresponding to P, satisfies $X_P = \mathbb{P}^n$, an α-invariant of $\frac{1}{n+1}$ can be achieved by taking a dot product between a point on P and a point on P° can be achieved. Otherwise, a polytope in n-dimensional space must have an α-invariant greater than or equal to $\frac{1}{n}$.*

In addition, we provide an explicit classification of the α-invariants for these varieties, showing a construction for each possibility.

Theorem 1.5. *Let X be a n-dimensional K-semistable toric smooth Fano variety. The possible values of $\alpha(X)$ for X are exactly $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}$.*

I can also explicitly pinpoint the varieties that satisfy $\alpha(X) = \frac{1}{2}$, showing that it exactly corresponds to polytopes that are *centrally symmetric*, which means it is the same upon reflection about the origin. This is a powerful necessary and sufficient condition that connects an important property of polytopes to a particular value of the α-invariant.

Theorem 1.6. *Let P be a K-semistable smooth Fano polytope. Then, we have that $\alpha(P) = \frac{1}{2}$ if and only if P is centrally symmetric.*

I also perform computations based on the *Picard number* $\mathrm{Pic}(X)$, or the rank of the Picard group of X. The Picard group denotes the group of isomorphism classes of line bundles that lie on X, and in the polytope formulation of toric varieties, it is n less than the number of vertices of the polytope. Casagrande [3] showed that $\mathrm{Pic}(X) \leq 2n$ if n is odd and $\mathrm{Pic}(X) \leq 2n-1$ if n is even. We prove the following theorem specifically with polytopes of small Picard number, connecting the Picard number to possible values of the α-invariant.

Theorem 1.7. *Let X be a K-semistable smooth toric Fano variety. Then, if $\mathrm{Pic}(X) = 1$, there is one unique polytope with $\alpha(X) = \frac{1}{n+1}$. If $\mathrm{Pic}(X) = 2$, there are $\left\lfloor \frac{n}{2} \right\rfloor$ different varieties corresponding to the α-invariants of*

$$\frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{\left\lceil \frac{n}{2} \right\rceil}.$$

1.1 Outline of the paper

We cover the background of toric varieties and the use of polytopes in Section 2. In Section 3, we show the main result: that Jiang's conjecture can be proven true assuming Ewald's conjecture. In Section 4 we provide examples of varieties that satisfy certain α-invariants and we prove Theorem 1.5 and Theorem 1.6, classifying the attainable values for the α-invariant in terms of polytopes. In Section 5, we prove Theorem 1.7, which gives a result on the α-invariant of varieties with small Picard number.

2 Preliminaries

Throughout this paper, we will be working over the complex field $\mathbb{C}$. An *affine variety* on $\mathbb{C}^n$ is defined by the solution set of the polynomial equations $f_1 = f_2 = \cdots = f_k = 0$, for some k. A *projective* variety is similarly defined over the n-dimensional projective space $\mathbb{P}^n$, with homogeneous equations.

2.1 Toric Varieties

I will begin by giving a brief introduction to toric varieties. Toric varieties allow for the overlap between algebraic geometry and the geometry of polyhedra. Just as affine varieties fitting together in order to construct algebraic varieties, cones are fit together to form fans and polytopes when considering the toric case.

Definition 2.1. A *toric* variety X is an algebraic variety in which there exists an open dense embedding from $(\mathbb{C}^\times)^n$ to X and where the action from $(\mathbb{C}^\times)^n$ to $(\mathbb{C}^\times)^n$ extends to a morphism from $(\mathbb{C}^\times)^n$ to X.

One can visualize toric varieties as those that can contain a torus inside it. The way we will work with affine toric varieties is by using cones.

Definition 2.2. A *cone* $\sigma \in \mathbb{R}^n$ consists of the set of points

$$\{\lambda_1\mathbf{u}_1 + \cdots + \lambda_l\mathbf{u}_l \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_l \geq 0\},$$

where l is a positive integer and $\mathbf{u_1}, \ldots, \mathbf{u}_l$ are lattice points in $\mathbb{Z}^n$. A *face* of a cone is the intersection of σ with some linear form $\{l = 0\}$. An *edge* of a cone is a face with dimension 1. A cone is called *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$.

A cone is *smooth* if it is generated by a basis of points in $\mathbb{Z}^n$, and *simplicial* if it is generated by points in $\mathbb{R}^n$. The *dual* of a cone σ is the set $\sigma^\vee = \{\mathbf{m} \in \mathbb{R}^n \mid \mathbf{m} \cdot \mathbf{u} \geq 0,\ \forall \mathbf{u} \in \sigma\}$, where $\mathbf{m} \cdot \mathbf{u}$ denotes the dot product.

Proposition 2.1 (Correspondence between cones and affine toric varieties). *Given a strongly convex cone $\sigma \in \mathbb{R}^n$, $\sigma^\vee \cap \mathbb{Z}^n$ is a finitely generated lattice. Then, the affine toric variety associated with a cone σ is $U_\sigma = \text{Spec } \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$.*

Here, Spec(R) denotes the spectrum of the commutative ring R. This shows that the building blocks of toric varieties, affine toric varieties, correspond directly to cones, which will be used to build polytopes.

Example 2.1. I present an example of a simplicial cone that is not smooth. Consider the cone $\sigma \in \mathbb{C}^2$ generated by e_2 and $2e_1 - e_2$. It generates all points with $\mathbb{R}$ coefficients, but not $\mathbb{Z}$ coefficients, because e_1 cannot be formed as a linear combination of e_2 and $2e_1 - e_2$ with integer coefficients.

In order to find U_σ, we take the dual of σ and consider that Spec $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ is generated by the points $(1, 0)$, $(1, 1)$, and $(1, 2)$, which correspond to the polynomials x, xy, and xy^2, respectively. Then,

$$U_\sigma = \text{Spec } \mathbb{C}[x, xy, xy^2] = \text{Spec } \mathbb{C}[u, v, w]/(v^2 - uw).$$

It is well known that this has a singularity that is not smooth; it is known as the ordinary double point A_1 [5].

This directly connects cones to affine toric varieties. It can be seen that just like affine toric varieties can be glued together to form toric varieties, cones can be glued together to form objects called fans. Properties that apply to cones, such as smoothness, will also still hold when glued together.

Definition 2.3. A *fan* Σ is a finite collection of cones in $\mathbb{R}^n$ such that each of the cones is strongly convex and rational, and the intersection of two cones is a face of both cones.

There is a variety X_Σ created by gluing together the affine varieties U_σ for all of the cones σ along their intersections. Every fan corresponds to a normal toric variety, and every normal toric variety has a corresponding fan.

Cones are shown to directly correspond to toric varieties by the Orbit-Cone Correspondence [4], which states that there is a bijective correspondence between cones on a fan Σ and the torus actions, or orbits, on its toric variety X_Σ.

2.2 Polytopes

In this section, we introduce *polytopes*, the central combinatorial structures analyzed in this paper. This polytope construction is equivalent to using the fan notation, and it directly allows us to perform calculations on toric varieties using polytope geometry. As such, one can think of every polytopes as directly corresponding to a toric variety.

Definition 2.4 ([4])**.** A *lattice polytope* P is the convex hull of a finite set of points S in $\mathbb{Z}^n$, or

$$P = \mathrm{Conv}(S) = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0, \sum_{u \in S} \lambda_u = 1 \right\}.$$

We will be considering polytopes in dimension n, which means there is no affine subspace of $\mathbb{Z}^n$ that contains P. A *face* of the polytope is defined as the nonempty intersection of P and a hypersurface H such that P is contained in one of the closed half-spaces defined by H; in other words, a certain part of the outside of P, much like vertices, edges, and faces of polyhedra are considered in 3 dimensions.

Faces of dimension 0 are called *vertices*, faces of dimension 1 are called *edges*, and faces of dimension $n-1$ are called *facets*. We assume that the polytopes considered contain the origin as an interior point; in fact, because of the smooth Fano condition, the origin is the only interior point of P. We denote the vertex set of P as $\mathcal{V}(P)$; this set is important because our computations of the α-invariant center around the vertices of a polytope.

Throughout this paper, we will denote points in $\mathbb{R}^n$ as $(x_1, x_2, \ldots, x_n)$. As we am considering lattice polytopes, these points will also always lie in $\mathbb{Z}^n$.

Similarly to how cones have duals, we can define the dual polytope of P, in terms of the intersection of half spaces determined by the vertices of P.

Definition 2.5. The *dual polytope* P° to a polytope P is defined by

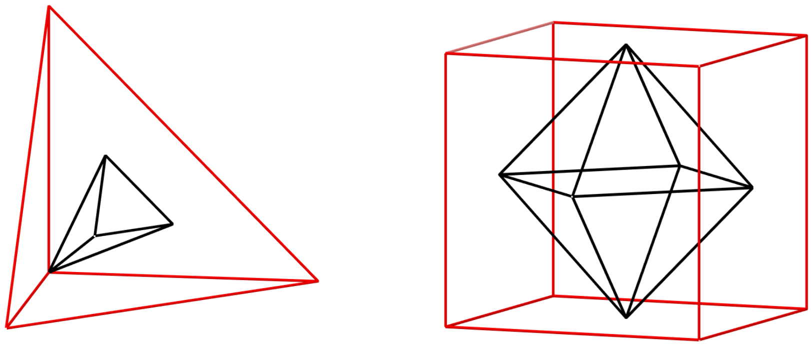
$$P^\circ = \{u \in \mathbb{R}^n \mid \langle u, v \rangle \geq -1 \quad \forall v \in \mathcal{V}(P)\}.$$

Each point on P corresponds to a hyperplane on P°, and vice versa; hence the duality. It is also possible to confirm that the dual of P° is P; in other words, the dual of the dual is the original polytope. The dual polytope will be necessary to convert the Fano condition to polytopes and also in order to define the α-invariant using polytope geometry.

Example 2.2. I give a few examples of toric varieties and their corresponding polytopes and dual polytopes.

1. Projective space $\mathbb{P}^n$ corresponds to the polytope with vertices at the point $(-1, -1, \ldots, -1)$ and the standard basis points e_i, which consists of only a 1 on the ith coordinate. The dual polytope has one vertex at $(-1, -1, \ldots, -1)$ and other vertices at the n distinct permutations of $(n, -1, -1, \ldots, -1)$.

2. The variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, consists of the product of n copies of the projective line $\mathbb{P}^1$. The polytope corresponding to this variety has vertices at the standard basis points e_i as well as their negatives $-e_i$. The dual polytope is thus defined by the hyperplanes $x_i = \pm 1$ for all coordinates x_i, and it has vertices at the points with all coordinates ± 1.



(a) $\mathbb{P}^3$ **(b)** $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Figure 1: The polytopes of these two varieties (black) along with their dual polytopes (red)

Analyzing the relationship between polytopes and their duals will be key to evaluating the α-invariant. All properties we consider, including K-semistability and smooth Fano, hold true for all equivalent polytopes corresponding to a particular variety.

I can provide a combinatorial formulation of those other properties seen in varieties and translate them to polytopes. A Fano toric variety corresponds to a Fano polytope, which we will define. The same holds for the smooth and K-semistable conditions we assume of varieties.

Definition 2.6. A *reflexive* lattice polytope is one whose dual is also a lattice polytope.

It turns out that Fano varieties correspond to reflexive polytopes, so we can call these Fano polytopes. When applying the smooth condition, we have that a *smooth Fano* polytope in $\mathbb{Z}^n$ is one in which the vertices of every facet form an integral basis for $\mathbb{Z}^n$.

It is also important to note that the representations shown in Figure 1 are not the only possible representations for those toric varieties. One toric variety can correspond to multiple different polytopes, all of which can be mapped to each other via a linear transformation with determinant ± 1. This is because a unimodular transformation preserves the volume of the polytope, as well as the volume of each individual *simplex*, or polytope between the origin and the vertices of a particular facet. A simplex is the convex hull of $n+1$ points, and it turns out that every simplex with a facet and the origin in a smooth Fano polytope must have minimal area.

In a smooth Fano polytope, every single simplex has the same volume, which is the same as the volume of the standard simplex consisting of the standard basis points e_i as well as the origin.

As any integral basis can be mapped to the standard basis via a unimodular transformation, we can apply the following proposition:

Proposition 2.2. *Every smooth Fano polytope can be mapped into one that contains the standard basis, defined by* $e_1, e_2, \ldots, e_n$.

Due to this proposition, we make the assumption that all of the polytopes we consider contain the standard basis points e_i.

In fact, Øbro showed that all smooth Fano polytopes can be bounded in the following space upon making this type of transformation.

Theorem 2.3 (Øbro [11]). *There exists an embedding of the vertex set of any n-dimensional smooth Fano polytope P into $\mathcal{W}_n$, a set of lattice points, such that the points in $\mathcal{W}_n$ are primitive lattice points $(a_1, a_2, \ldots, a_n)$ and $\sum_i a_i = a$, then $-n \leq a \leq -1$ and:*

1. *If $a = 1$, $0 \leq a_i \leq 1$.*
2. *If $a = 0$, $-1 \leq a_i \leq n-1$.*
3. *If $a < 0$, $a \leq a_i \leq n + a$.*

for all $1 \leq i \leq n$.

This gives a condition and a bound on all smooth toric Fano varieties, showing that there is, in fact, a finite number of equivalence classes of smooth Fano varieties.

The last condition we consider in varieties is K-semistability, which further narrows down the range of possibilities. K-semistability had not been significantly addressed for toric varieties, and not in the context of the α-invariant at all.

Theorem 2.4 ([7]). *A polytope P corresponds to a K-semistable toric Fano variety if and only if its barycenter, or center of mass, is the origin.*

The computation of the barycenter of general polytopes is difficult, so we will provide the following proposition that applies to smooth Fano polytopes, which allows us to evaluate K-semistability purely based on the vertices. It is the smoothness that allows us to translate K-semistability into a very powerful condition.

Proposition 2.5. *For K-semistable smooth Fano polytopes, the average of all vertices is the origin, or the barycenter.*

Remark. In general, it is not true that the average of the vertices of a polytope is the same point as its barycenter.

I give an example of these polytopes in the 2-dimensional case.

Example 2.3. There are 16 equivalence classes of reflexive polygons, corresponding to Fano toric varieties in 2 dimensions. They are shown in the Figure 2. Of these, 5 are smooth Fano; they are labeled 3, 4a, 4b, 5a, and 6a. These are the polytopes such that no edge has more than 2 lattice points. Three of the smooth Fano polytopes are also K-semistable, and they are labeled 3, 4a, and 6a.

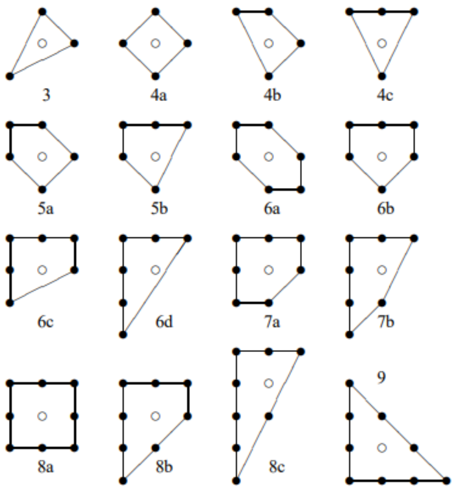


Figure 2: 2-dimensional reflexive polytopes [4]

I can use the fact that K-semistability and smooth Fano are both strong conditions, which allows us to analyze the polytopes using combinatorial tools more easily. It turns out that there are always a finite number of reflexive polytopes, because the reflexivity ensures that the volume of these polytopes must stay somewhat small.

Another important concept important to varieties is the *Picard number*, which very conveniently translates into the combinatorial formulation. It turns out that $\mathrm{Pic}(X) = |\mathcal{V}(P)| - n$. We will investigate the α-invariants of polytopes with small Picard number using a combinatorial bounding argument.

Finally, an important class of polytopes is those that are *centrally symmetric*, which means that reflecting all points across the origin gives the same polytope. These polytopes were studied by Ewald [6] and Øbro [11].

This completes the description of the polytopes we will consider. Now, we define the α-invariant, our central topic of study, in terms of polytopes and the tools described in this section. This requires using the *dot product* between points on P and P° in order to assign a numerical value to the polytopes considered. If we have a point $\mathbf{u} = (u_1, u_2, \cdot, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Proposition 2.6 (α-invariants using polytopes)**.** *Given a K-semistable smooth toric Fano polytope P corresponding to a variety X_P and its dual polytope P°, $\alpha(X_P)$ is the smallest possible value of $\frac{1}{\mathbf{u}\cdot\mathbf{v}+1}$, where $\mathbf{u}$ is a vertex of P and $\mathbf{v}$ is a vertex of P°.*

Since each polytope corresponds to exactly one toric variety, we will denote $\alpha(X_P)$ as simply $\alpha(P)$ on the polytope.

This allows us to find the α-invariant purely by considering the vertices present in P and P°. It is also true that under a unimodular transformation, the α-invariant stays the same, which means our previous assumption transforming the polytopes considered into the standard basis vectors still holds with the α-invariant. This is because the dot products we use to define the α-invariant are not altered upon such a transformation. Now we can restate Conjecture 1.2 in the toric case, by translating conditions on varieties into conditions on polytopes.

Conjecture 2.7. *Given a K-semistable smooth toric Fano polytope P, $\alpha(P) \geq \frac{1}{n}$ unless $\alpha(P) = \frac{1}{n+1}$, in which P must be the polytope corresponding to $\mathbb{P}^n$.*

3 Jiang's Gap Conjecture

In this section, we will assume the following version of Ewald's conjecture in order to prove Theorem 1.4.

Conjecture 3.1 (Weak Ewald [6]). *Up to a unimodular transformation, all vertices of a Fano polytope have coordinates greater than or equal to -1, and the polytope also contains all of the standard basis vectors e_i.*

However, Ewald's conjecture is not needed for the complete proof; the first half is not dependent on Ewald's conjecture.

Now, using these lemmas, we can prove Jiang's conjecture in the toric case, in the following form.

Theorem 3.2. *Suppose that Conjecture 3.1 is true. If X_P, the variety corresponding to a polytope P, gives $X_P = \mathbb{P}^n$, we can achieve a dot product of n between a point on P and a point on P°. Otherwise, a polytope in n dimensional space must have a dot product of at most $n-1$ between the two polytopes.*

Proof. Consider the point on P which the dot product is the maximum. Take a unimodular transformation that takes this point to $e_1 = (1, 0, \ldots, 0)$, and one of the facets it lies on as $e_2, e_3, \ldots, e_n$. Note that this means the rest of the points on P add up to $(-1, -1, \ldots, -1)$ due to K-semistability.

The point on P° with the greatest first coordinate must have first coordinate at least n. Call this point Z. We want to prove that this only works if P is the polytope corresponding to $\mathbb{P}^n$.

The steps of the proof are as follows.

1. We claim that all points on P that are not e_i must have a dot product of 0 or -1 with Z. The only way this works is if the sum of the nonnegative coordinates of Z is at least n. However, we only need to consider when the first coordinate is n, because it provides the greatest dot product.

2. We prove using Ewald's conjecture that the only possibility for Z is $(n, -1, -1, \ldots, -1)$; none of its coordinates can be 0.

3. We prove that if $Z = (n, -1, -1, \ldots, -1)$, P must be the polytope corresponding to $\mathbb{P}^n$.

I begin by noting that because we transformed a facet of P to contain the standard basis vectors e_i, this provides a constraint on the location of the points on P°. Recall that the definition of the dual polytope means that each point on P corresponds to a facet on P°, in a way such that the dot product of all points on P with P° are at least -1.

This means that each point e_i on P defines the hyperplane $x_i \geq -1$. As such, points on the dual polytope cannot have coordinates less than -1.

This means we can express Z as $Z = (a_1, a_2, \ldots, a_k, -1, \ldots, -1, -1)$ for some positive integer k, where we can assume without loss of generality that the -1 coordinates are at the end and $a_i \geq 0$ for all i.

Consider the point $S = (-1, -1, \ldots, -1)$. We have that $S \cdot Z = n - k - \sum_{i=1}^{k} a_i$. Also, note that because of the K-semistable condition, the sum of the non basis points is S. Thus, taking the dot product with Z, we have

$$\sum_{\mathbf{u} \neq e_i, \mathbf{u} \in P} \mathbf{u} \cdot Z = S \cdot Z.$$

It turns out that we can complete step 1 of the proof with the following lemma:

Lemma 3.3. *Let P be a K-semistable smooth Fano polytope. Given any point on P°, the sum of the nonnegative coordinates of that point are at most n.*

Proof. Consider a point Y on P°. we will express Y as $(b_1, b_2, \ldots, b_k, -1, \ldots, -1, -1)$ where we can assume without loss of generality that the -1 coordinates are at the end and $b_i \geq 0$ for all i.

Since Y is on P°, because of the duality condition, it creates a half space in P, defined by $Y \cdot p \geq -1$ for all points p, or

$$b_1 x_1 + b_2 x_2 + \cdots + b_k x_k - x_{k+1} - \cdots - x_n \geq -1.$$

There are n points that lie on the hyperplane, which is the equality case of the above inequality, because Y corresponds to a facet on P which must have n vertices. However, this facet also contains $n - k$ of the basis vectors, which correspond to e_t where coordinate t of Y is -1. This means that at most k of these vectors contribute to the sum of the dot products of Y.

Since we know that the dot product of Y with any point in P is at least -1, and there are at most k non basis points on p that give a dot product of -1, we find that

$$\sum_{\mathbf{u} \neq e_i, \mathbf{u} \in P} u \cdot Y \geq -k.$$

Now, we use the equation $\sum_{\mathbf{u} \neq e_i, \mathbf{u} \in P} \mathbf{u} \cdot Y = S \cdot Y$ to get

$$S \cdot Y \geq -k \implies n - k - \sum_{i=1}^{k} b_i \geq -k \implies n \geq \sum_{i=1}^{k} b_i.$$

This is the inequality we wanted to prove; b_i are the nonnegative coordinates of of Y, and we have proved that their sum is at most n. □

This lemma can be directly applied to Z, which we put in the form

$(a_1, a_2, \ldots, a_k, -1, -1, \ldots, -1)$. However, we also assume that a_1 is at least n, in order to get a dot product of n. This means that the only possibility is $a_1 = n$ and $a_2 = a_3 = \cdots = a_k = 0$. Otherwise, the dot product could be at most $n - 1$.

This also means that the inequality $\sum_{\mathbf{u} \in P} u \cdot Z \geq -k$ is an equality, which implies that there are no $\mathbf{u} \neq e_i$ in P such that $\mathbf{u} \cdot Z > 0$, because exactly k of them are -1.

Thus, we have shown that Z is of the form $(n, 0, 0, \ldots, 0, -1, -1, \ldots, -1)$ and that for all non basis points $\mathbf{u} \in P$, the dot product $Z \cdot \mathbf{u}$ is either 0 or -1. This finishes part 1 of the proof.

For the rest of this proof, we will assume the Conjecture 3.1, or that all coordinates x_i satisfy $x_i \geq -1$.

Now, we will solve part 2 of the proof. we know that all points other than the e_i have a dot product of 0 or -1 with Z, and the vector sum of these points is S, with all negative coordinates. This means that there must be at least one point $\mathbf{w}$ on P which has first coordinate -1, since we are assuming all coordinates have magnitude at most 1.

Then, $\mathbf{w} \cdot Z = -n + C_2 + C_3 + \cdots + C_n$, where C_i denotes the dot product achieved from the ith coordinate. However, due to Ewald's conjecture, $C_i \leq 1$ for all i, because the respective term in Z is either -1 or 0, and the respective coordinate in $\mathbf{w}$ is at least -1. Now, we have that

$$-n + C_1 + C_2 + \cdots + C_{n-1} \leq -n + (n-1) = -1$$

The only way we can get a value of 0 or -1 here is in the equality case, where $C_i = 1$ for all i. This means that $Z = (n, -1, -1, \ldots, -1)$, and $\mathbf{w} = (-1, -1, \ldots, -1)$.

Finally, we will prove the following lemma, which gives step 3 of the proof.

Lemma 3.4. *If $Z = (n, -1, -1, \ldots, -1) \in P^\circ$, the only possible polytope is the one corresponding $\mathbb{P}^n$. In other words, the only points on the polytope are the e_i and the point $S = (-1, -1, \ldots, -1)$.*

Proof. Assume otherwise, so P is not the polytope corresponding to $\mathbb{P}^n$. First, we prove that S must lie on the polytope P. This can be done by considering the facet on P that corresponds to the dual of the point Z, which must have n vertices. Z defines a hyperplane with equation

$$nx_1 - x_2 - x_3 - \cdots - x_n = -1.$$

Note that this facet must already contain the $n-1$ points $e_2, e_3, \ldots, e_n$. The first coordinate of the last vertex must be -1, so that this facet can be an integral basis, since the first coordinate of all other points is 0, and the rest of the points are already connected to e_1, with first coordinate 1.

This means that this vertex must satisfy $-n - \sum_{i=2}^{n} x_i = -1$, which means that $\sum_{i=2}^{n} x_i = 1 - n$. But each $x_i \geq -1$ by Ewald's conjecture, so

$$\sum_{i=2}^{n} x_i \geq -(n-1) = 1 - n,$$

with equality only when each $x_i = -1$. Thus, the last vertex on this facet must be $S = (-1, -1, \ldots, -1)$.

Now, we have proved that P must contain S and the standard basis points e_i. This means it contains at least one other point, because it is distinct from the polytope for $\mathbb{P}^n$.

Consider a vertex $B = (b_1, b_2, \ldots, b_n) \in P$, such that B is not e_i or S. we showed the dot product of any vertex not equal to e_i on P and Z is 0 or -1 in Lemma 3.3. However, the product cannot be -1, because there are already n points with dot product -1 with Z on P; namely, the facet consisting of S, e_2, $e_3, \ldots, e_n$.

This means that

$$B \cdot Z = nb_1 - b_2 - b_3 - \cdots - b_n = 0.$$

Adding $\sum b_i$ to both sides gives $(n+1)b_1 = \sum b_i$, so $\sum b_i \equiv 0 \bmod n+1$. However, we know that $-n \leq \sum b_i \leq 1$, so the only possibility is $b_1 = 0$. This also implies that $\sum_{i=2}^{n} b_i = 0$.

Now, notice that it is also true that for all coordinates, we have $b_i \geq 1$, and there must be at least one negative b_i, so one of the coordinates in B must be -1. Without loss of generality, let this coordinate be b_2. Also, there must be another odd coordinate, since the total sum is even, so without loss of generality, let b_3 be odd. We can make these assumptions because all coordinates are interchangeable by symmetry.

Consider the following hyperplane: $x_1 = 0$, $x_3 + x_4 + \cdots + x_n = 1$. It has dimension $n-1$, and it contains the following $n-1$ points: B, e_3, $e_4, \ldots, e_n$. This means that no other points in P can lie on this hyperplane. This also means that B is connected to $e_3, e_4, \ldots, e_n$.

B is also connected to e_1 and S, as e_1 and S are the only points with nonzero first coordinates of 1 and -1, respectively. In fact, this means that e_1 and S are both connected to all other points on P.

Now, we find that B is connected to n other points, forming n different facets. Consider the facet formed by e_1, e_4, $e_5, \ldots, e_n$, S, and B. Since we assumed that b_3 was odd, then $b_2 + b_3$ must be even. we can see here that in all points on this facet, $x_2 + x_3$ is even. This means that the points on this facet are not a basis for $\mathbb{Z}^n$.

Thus, we have arrived at a contradiction, since P is smooth Fano, and we have finished the proof using the fact that this facet must exist and that its vertices are not a basis. □

This concludes the proof of Theorem 3.2. Given a K-semistable smooth Fano polytope, we can achieve an α-invariant of $\frac{1}{n+1}$ if and only if the polytope P corresponds to $\mathbb{P}^n$, and for all other polytopes, the α-invariant is at least $\frac{1}{n}$.

□

4 Classification of the α-invariant

In this section, we will not assume Ewald's Conjecture. we still adopt the notion that all polytopes contain the standard basis, which works because it is possible to map any facet to the standard basis in a smooth Fano polytope.

First, we will prove a theorem that evaluates the α-invariant for a certain type of polytopes with no positive coordinates outside of the standard basis e_i.

Theorem 4.1. *Consider a partition c_j of n, where $\sum_{1\leq j\leq m} c_j = n$ and m is the number groups in the partition. Then, consider a set of points D_j where point D_j has c_j coordinates of -1, and no two D_j both have the same coordinate as -1. Take the polytope P with the standard basis e_i as well as these points D_j. Then, P is a K-semistable smooth Fano polytope that satisfies $\alpha(P) = \frac{1}{\max(c_j)+1}$, where the sum is taken over the m groups of the partition.*

Proof. I begin with an example to demonstrate this construction. Consider $n = 7$ and let the partition be $c = (3, 2, 1, 1)$. Then, we would have $D_1 = (-1, -1, -1, 0, 0, 0, 0)$, $D_2 = (0, 0, 0, -1, -1, 0, 0)$, $D_3 = (0, 0, 0, 0, 0, -1, 0)$, $D_4 = (0, 0, 0, 0, 0, 0, -1)$. This polytope satisfies $\alpha(P) = \frac{1}{4}$ because the point $(3, -1, -1, 2, -1, 1, 1)$ would lie on the dual P° and it is possible to prove that no point with a coordinate greater than 3 does.

Now, we proceed to the proof that this construction always yields a K-semistable smooth Fano polytope. Clearly, it is always smooth Fano; all of the D_j sum to $S = (-1, -1, \ldots, -1)$. In order to show that it is smooth Fano, we need to show that each facet is an integral basis for $\mathbb{Z}^n$. It actually turns out that we can show that any set of n linearly independent points from this set form an integral basis.

I can do this by considering each individual D_j as well as the standard basis vectors with coordinates in D_i, because we partitioned the coordinates into distinct groups. Without loss of generality, take $D_1 = (-1, -1, \ldots, -1, 0, 0, \ldots, 0)$, with t of the first coordinates being -1. Then, if we were to take a set of N linearly independent points from P, D_1 can either be part of the set or not. We consider both cases.

If D_1 is not part of the set, then $e_1, e_2, \ldots, e_t$ must all be in the set, because these are the only remaining points with a nonzero coordinate at their respective values. Clearly, this forms an integral basis for this copy of $\mathbb{Z}^t$ that lies inside $\mathbb{Z}^n$.

If D_1 is part of the set, then we choose any $t-1$ of the basis points $e_1, e_2, \ldots, e_t$. It is easy to see that this collection of t points also forms an integral basis for the first t coordinates.

Thus, we can put together the coordinates for all D_j, finding that each individual set of points is distinct and is able to generate all lattice points spanned by those coordinates. Since this proves that any linear combination of points in this construction is an integral basis, the construction thus holds and these are valid K-semistable smooth Fano polytopes.

All that is left to do at this point is evaluate their α-invariants.

Consider the half-spaces in the dual defined by each of the points on the polytope. All of the e_i points guarantee that the dual points all must have every coordinate at least -1. Each D_i corresponds to a group of coordinates, and the sum of the coordinates in its group must be at most 1, because a point $(0, 0, \ldots, 0, -1, -1, \ldots, -1, 0, 0, \ldots, 0)$ with the coordinates a to b being -1 gives the equation

$$-x_a - x_{a+1} - \cdots - x_b \geq -1 \implies \sum_a^b x_i \leq 1.$$

Since each of the individual coordinates is at least -1, we can compute that

$$\sum_a^b x_i \leq 1 \implies x_a \leq 1 - \sum_{a+1}^b x_i \leq 1 - (b-a) \cdot (-1) = b - a + 1.$$

This implies that the maximum possible coordinate on P° from each set of vertices from D_i is exactly the number of nonzero coordinates in D_j, which is $b - a + 1$. This means that taking the dot product of any of the e_j gives a maximum equal to $\max_{1 \leq j \leq m} c_j$.

Taking the dot product with one of the D_j points is similar. Because D_j has coordinates only 0 and -1, and because we know all points in the dual have every coordinate at least -1, the maximum possible dot product between D_j and a point on the dual is the number of -1 coordinates that D_j has, or c_j.

Thus, the overall maximum possible dot product for our constructed polytope is $\max_{1\leq j\leq m} c_j$, as desired. $\square$

Now, using Theorem 4.1 as a key example of the α-invariants that can be achieved, we will prove Theorem 1.5, which states that no other values of the α-invariant are possible.

Proof of Theorem 1.5. Once again, let P be the polytope corresponding to X, a K-semistable smooth toric Fano variety.

I want to show that the maximum possible value of the dot product between a point on P and a point on P° can be any positive integer less than or equal to n. We will first show that these are the only possible numbers that can be achieved, and then we will show a construction for all of these dot products.

Again, we assume without loss of generality that the largest possible dot product corresponds to $e_1 = (1, 0, \ldots, 0)$ on P. This means we want to find the point on P° with the largest first coordinate. Because the dual polytope is n-dimensional and it must also contain the origin as an interior point, it must have a point with first coordinate greater than 0, which implies that the dot product must be greater than 0.

In order to show that the dot product must be at most n, we use Lemma 3.3, which showed that the sum of the nonnegative coordinates for any point on P° is at most n. This means that each individual coordinate must be at most n as well, which proves that n is the maximum.

Since we are considering only integral points, it is clearly true that the only possible values of the dot product are thus $1, 2, \ldots, n$.

Now, because of Theorem 4.1, we know that there is an explicit calculation for all of these possible α-invariants, because there exist partitions of n for which the maximum number is any integer between 1 and n, inclusive. Thus, the classification is complete. $\square$

Now, we take a closer look at a larger case of the α-invariant: when it is $\frac{1}{2}$. We prove Theorem 1.6, connecting the centrally symmetric polytopes to this particular value of the α-invariant.

Proof of Theorem 1.6. First, we prove that all K-semistable smooth Fano polytopes P that are centrally symmetric satisfy $\alpha(P) = \frac{1}{2}$.

Let $\mathbf{u}$ be the point in the vertex set $\mathcal{V}(P)$ that corresponds to the largest possible dot product with a point in P°. Since the polytope is centrally symmetric, $-\mathbf{u}$ also lies in $\mathcal{V}(P)$. What this means is that for all points $\mathbf{v}$ on P°, we have that $(-\mathbf{u}) \cdot \mathbf{v} \geq -1$, by the definition of the dual. However, this rearranges to be $\mathbf{u} \cdot \mathbf{v} \leq 1$, which means the dot product of $\mathbf{u}$ with any other point on the polytope is at most 1.

I also know that the sum of all vertices in the polytope is 0, which means

$$u \cdot \sum_{\mathbf{v} \in \mathcal{V}(P^\circ)} \mathbf{v} = 0.$$

Not every term in the sum can dot to 0 with $\mathbf{u}$, because the polytope is n-dimensional, which means not all lie in a hyperplane perpendicular to $\mathbf{u}$. This implies that at least one of the points in P° have a positive dot product with $\mathbf{u}$. Since the dot product must be an integer and it cannot exceed 1, it has to be 1. Thus, $\alpha(P) = \frac{1}{2}$.

Now, we prove the opposite direction. We first assume that P is a K-semistable smooth Fano polytope P such that $\alpha(P) = \frac{1}{2}$. Now, consider a point $\mathbf{u}$ in the vertex set $\mathcal{V}(P)$. We will prove that $-\mathbf{u}$ must also belong to the vertex set.

I know that $\mathbf{u} \in P$, which implies there is a facet defined by $\mathbf{u}$ in P° which has at least n points. All points $\mathbf{v}$ on this facet satisfy $\mathbf{u} \cdot \mathbf{v} = -1$.

Because $\alpha(P) = \frac{1}{2}$, the maximum possible dot product between u and a point on P° is 1. Since we know that the polytope is K-semistable, we know that

$$u \cdot \sum_{\mathbf{v} \in \mathcal{V}(P^\circ)} \mathbf{v} = 0,$$

which is the same equation used in the first part of the proof. This is composed of dot products of only -1, 0, and 1. This means that there is the same number of points which have a product of -1 and 1 with $\mathbf{u}$, so there are at least n points on the hyperplane defined by $\mathbf{u} \cdot x = 1$ for points $x = (x_1, x_2, \ldots, x_n)$.

However, if there are more than n points on this hypersurface, it corresponds to a facet, which translates back to the point $-\mathbf{u}$, because the hypersurface $\mathbf{u} \cdot x = 1$ is dual to the point $-\mathbf{u}$. This proves that $-\mathbf{u}$ must be on P. Since $\mathbf{u}$ is arbitrarily defined out of all of the vertices of P, this holds for any choice of vertex, which means that P is centrally symmetric, as desired.

Thus, both sides of the equivalence are shown, which means that the conditions $\alpha(P) = \frac{1}{2}$ and centrally symmetric are exactly equivalent for K-semistable smooth Fano polytopes. $\square$

5 α-invariants on varieties with small Picard number

In this section, we prove Theorem 1.7, regarding the polytopes with Picard number 1 or 2. This essentially proves that all possible polytopes with Picard number 1 or 2 are listed in Theorem 4.1.

Proof of Theorem 1.7. I first begin with polytopes with Picard number 1, which means that they have exactly $n+1$ vertices. Since n of them are the standard basis vectors e_i, the last one must be $S = (-1, -1, \ldots, -1)$ because the polytopes are K-semistable. This means that there is only one possible polytope in this case.

Now, we consider polytopes with Picard number 2. This means that there are 2 vertices that are not standard basis vectors, and they sum to S due to K-semistability. Call these points A and B.

First, we prove that none of the coordinates for any of these points can be positive or less than -1. Assume otherwise for the sake of contradiction. This means that we can assume without loss of generality that A has a coordinate, say, the first coordinate, that is less than -1. This would imply that the first coordinate of B is positive, since they sum to -1.

Now, consider the simplex formed by the n points A, e_2, $e_3, \ldots, e_n$. This simplex must exist because there is no other point on the polytope with a negative first coordinate that can connect to the $n-2$ dimensional face with e_2, $e_3, \ldots, e_n$. It must be part of two facets; one is with e_1, and the other one must be with A, because A has negative first coordinate.

The volume of this simplex, however, is greater than that of the standard simplex. Since A is a distance greater than 1 away from the hyperplane $x_1 = 0$ that the rest of the points lie on, it cannot be part of a valid polytope. Thus, we arrive at a contradiction, because this means that P cannot be smooth Fano. Another way to state this is to reason that the vertices of the simplex do not form an integral basis, because only multiples of the first coordinate, which is not 1, would be obtained for the first coordinate of any linear combination.

This means we have proved that all coordinates of points that are not e_i in K-semistable smooth Fano polytopes with Picard number 2 must be -1 or 0. In order to find the remaining possibilities for point A, note that we can arrange the different coordinates without loss of generality, so A can just be $(-1, -1, \ldots, -1, 0, 0, \ldots, 0)$ for r different instances of -1 where $0 < r < n$. We can then immediately see that B must be $(0, 0, \ldots, 0, -1, -1, \ldots, -1)$, because A and B must add to S. Without loss of generality A has at least as many coordinates of -1 as B, which means we can calculate the α-invariant based on the number of such coordinates in A.

It turns out that these are all of the possibilities; there are exactly $\lfloor \frac{n}{2} \rfloor$ different varieties. These are also all a subset of the construction shown in Theorem 4.1. This means they are all K-semistable and smooth Fano, and we can confirm that they have α-invariants one greater than the maximum number of -1 coordinates in either A or B.

Thus, the theorem is proved and we have completely classified K-semistable smooth Fano polytopes with Picard numbers 1 and 2. □

6 Acknowledgments

We would like to thank the MIT PRIMES program for providing this opportunity to conduct research for high school students under the mentorship of graduate students and professors.

References

[1] S.-T. Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci. USA*, 74(5):1798-1799, 1977.

[2] T. Aubin. Equations du type monge-ampere sur la varietes kahleriennes compactes. *C. R. Acad. Sci. Paris*, 283:119–121, 1976.

[3] C. Casagrande. The number of vertices of a Fano polytope. *Ann. Inst. Fourier*, 56:121–130, 2004.

[4] H. S. D. Cox, J. Little. Toric varieties. *Grad. Stud. Math.*, 124, 2011.

[5] A. Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math.*, (2):131–163, 1979.

[6] G. Ewald. On the classification of toric Fano varieties. *Discrete Comput. Geom.*, (3):49–54, 1988.

[7] K. Fujita. K-stability of Fano manifolds with not small alpha invariants. *J. Inst. Math. Jussieu*, 18(3):519–530, 2019.

[8] C. Jiang. K-semistable Fano manifolds with the smallest alpha invariant. *Internat. J. Math.*, 28(06):1750044, 4 2017.

[9] G. Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(m) > 0$. *Invent. Math.*, (2):225–246, 1987.

[10] S. S. X.-X. Chen, S. Donaldson. Kähler-Einstein metrics and stability. *Int. Math. Res. Not.*, 8:2119–2125, 2014.

[11] M. Øbro. An algorithm for the classification of smooth Fano polytopes, 2007. arXiv:0704.0049.