On Subsets Sums and Thin Additive Bases

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Additive Bases

Let's start with an example: Suppose that B is the set of all odd integers. Clearly, every integer can be represented as a sum of at most 2 elements of B.

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Example

The set $\{1, 2, \ldots, m-1\} \cup \{m, 2m, 3m, \ldots\}$ is an additive basis of order 2, because every number can be written as $pm + r$.

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Example (Lagrange)

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Example (Goldbach)

If true, then the set of prime numbers is an additive basis of order 3.

Efficiency

Note that if B is an order k additive basis of $\mathbb{N}$, then $B \cap [n]$ is an order k additive basis of $[n]$. A simple calculation gives that $|B| \geq cn^{1/k}$.

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Given n, does an order k additive basis B of $[n]$ of size $\Theta(n^{1/k})$ exists?

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Main problem

This construction doesn't cover all elements uniformly. For example, $\sqrt{n} + 1$ is represented $\sqrt{n}/2$ times while $\sqrt{n} \cdot \sqrt{n} - 1$ or $\sqrt{n} \cdot \sqrt{n} - 2$ are both counted once.

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Theorem (Erdős and Tetali (1990))

Fix k, and let n be large. Then there exists a thin additive basis B of $\{1, 2, \dots, n\}$ with order k.

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Conjecture

There exists a thin additive basis B of n of order $k := k(n)$ for all $k = o(\log n / \log \log n)$.

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Our proof scheme goes more or less as follows:

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In order to overcome the limitations of step 2 we assign each number x a distinct probability p_x. This complicates our calculations so we will omit it from our talk.

Concentration

Let $Y(t_1, t_2, \ldots, t_n)$ be a polynomial of indicator random variables.

Main Idea

If the partial derivatives of a multivariate function of random variables are all small, then the quantity is strongly concentrated.

Theorem (Vu (2000))

For any positive constants $k, \alpha, \beta, \epsilon$, if a boolean polynomial Y is normal and homogeneous of degree k, $n/Q \geq E(Y) \geq Q \log n$ and $E(\partial_A(Y)) \leq n^{-\alpha}$ for all nonempty sets A of cardinality at most $k-1$, then

$$Pr(|Y - E(Y)| \geq \epsilon E(Y)) \leq n^{-\beta}.$$

Computational Support

Let $Y_{n,k}$ be the polynomial whose terms correspond to representations of n of order k. Observe that all partial derivatives take the form $Y_{n',k'}$ ($n' < n$ and $k' < k$).

- Pick each number with probability $q = e^{-2+c/k} n^{-1+1/k} k^2 (\log n)^{1/k}$

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- Thus, the Vu Inequality says our construction works with high probability.

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- Characterize the thinness of additive bases with large order.

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- Prove Erdős Turán conjecture.

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