On Base 3/2 and Greedy Partitioning of Integers

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Framework for Defining Bases

Exploding Dots (popularized by James Tanton)

- Boxes in a row — Digit Places
- If a box exceeds b dots, "explode" those b dots and replace them with a dots in the next box to the left

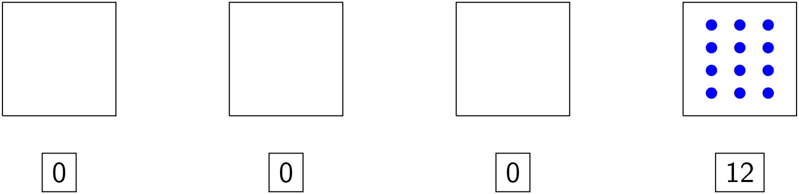
Notation

- a-b machine refers to exploding dots, where b dots become a dots in the next box to the left.

Integer Bases

- 1-n machine

Example: 2-3 Machine



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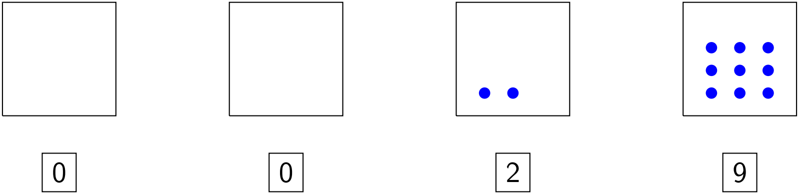

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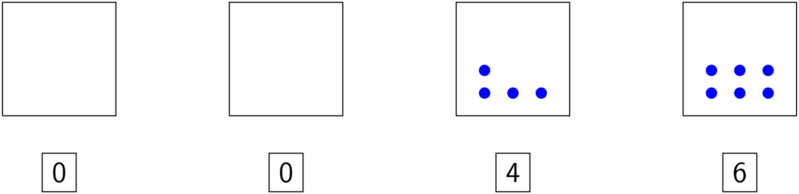

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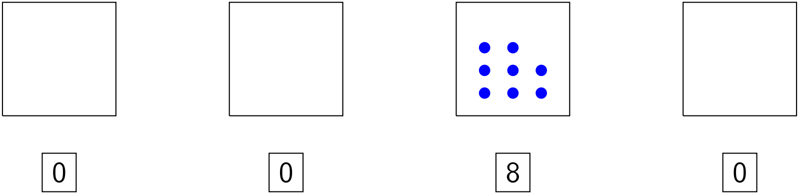

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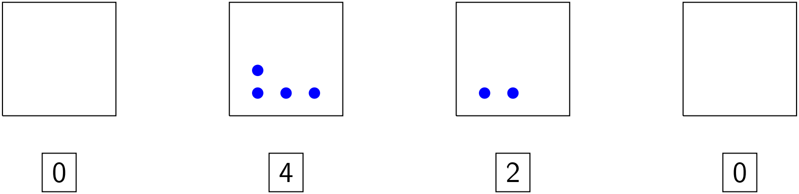
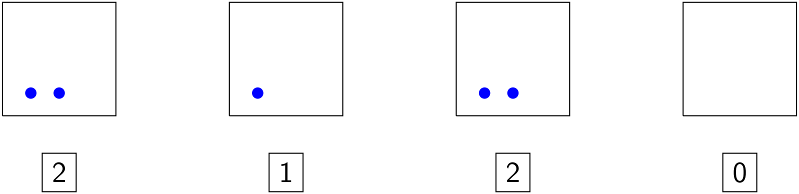


Example: 2-3 Machine



Even Numbers Base 3/2

Adding 2 to a base 2-3 number x has the following effect on the digits of x:

- Right of, and including the rightmost zero, each digit is reduced by 1 modulo 3. Each digit place to the left of the rightmost zero is unaffected.

Example:

$$2012021121 + 2$$

2012 | 021121+2

2012 | 210010

Even Numbers Base 3/2

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Example:

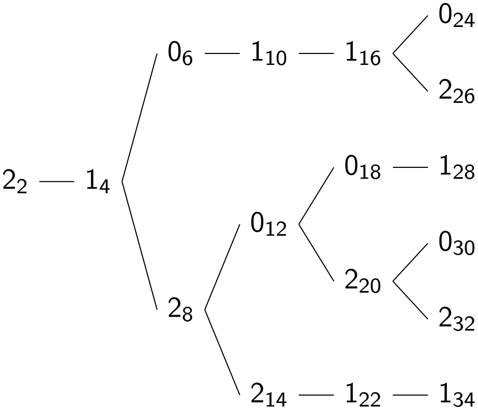
$$2012021121 + 2$$

2012   021121+2

2012   210010

0	2	4	6	8	10	12	14	16	18	...
0	2	21	210	212	2101	2120	2122	21011	21200	...

Tree within Base 3/2



Expanded Tree Sequence

Each sequence is created by adding 2 to the number preceding it base 2-3.

- 0, 2, 21, 210, 212, 2101, 2120, 2122, 21011, 21200, 21202, ...
- 1, 20, 22, 211, 2100, 2102, 2121, 21010, 21012, 21201, ...
- 10, 12, 201, 220, 222, 2111, 21000, 21002, 21021, 21210, ...
- 11, 200, 202, 221, 2110, 2112, 21001, 21020, 21022, ...
- 100, 102, 121, 2010, 2012, 2101, 2120, 2122, 21011, ...

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- 0, 2, 21, 210, 212, 2101, 2120, 2122, 21011, 21200, 21202, ...
- 1, 20, 22, 211, 2100, 2102, 2121, 21010, 21012, 21201, ...
- 10, 12, 201, 220, 222, 2111, 21000, 21002, 21021, 21210, ...
- 11, 200, 202, 221, 2110, 2112, 21001, 21020, 21022, ...
- 100, 102, 121, 2010, 2012, 2101, 2120, 2122, 21011, ...

Each first term is chosen to be the "smallest" not appearing in previous sequences. This procedure gives us the sequence of first terms to not have any 2.

Infinite Grid

As no term will be used twice in any of these sequences, we can form them into a grid.

0	1	10	11	100	101	110	111	...
2	20	12	200	102	120	112	2000	...
21	22	201	202	121	122	2001	2002	...
210	211	220	221	2010	2011	2020	2021	...
212	211	222	2110	2012	2200	2022	2210	...
2101	2100	2111	2112	2201	2202	2211	2212	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Base 3/2 Grid

3-free Sequences

Definition:

- A sequence is called *3-free* if it does not contain an arithmetic progression of length 3. *The Stanley sequence* is a lexicographically earliest 3-free sequence on the set of non-negative integers.

3-free Sequences (cont.)

Procedure:

- Start with 0, 1
- Skip 2 because 0, 1, 2
- Add on 3, 4
- Skip 5, because 1, 3, 5
- Skip 6, 7, 8
- Add on 9
- Continue picking the smallest not forming an arithmetic sequence.

$$S_0 \in Z^*$$

0	1	3	4	9	10	12	13	...

Greedy Partitioning

Form another 3-free sequence from the integers we passed over, excluding integers already in the first 3-free sequence.

2	5	6	11	14	15	18	29	...

$$S_n \in Z * - \sum_{k=0}^{n-1} S_k$$

And we continue creating new sequences in this fashion...
(J. Gerver, J. Propp, and J. Simpson, 1988)

Greedy Partitioning

Until we have another infinite grid...

0	1	3	4	9	10	12	13	...
2	5	6	11	14	15	18	29	...
7	8	16	17	19	20	34	35	...
21	22	24	25	48	49	51	52	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Greedily Partitioned Sequences Grid

Beginning to Form a Connection

Take all these 3-free sequences and represent them in base 3.

0	1	10	11	100	101	110	111	...
2	12	20	102	112	120	200	1002	...
21	22	121	122	201	202	1021	1022	...
210	211	220	221	1210	1211	1220	1221	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Greedily Partitioned Sequence Base 3 Grid

And this looks really similar to what we have before.
Conjecture by PRIMES STEP students, 2018!

Side by Side Look

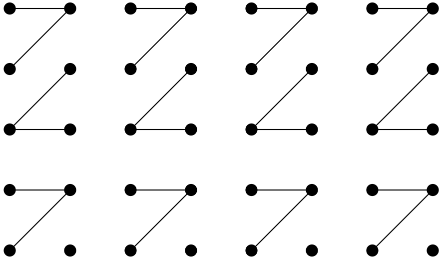
0	1	10	11	100	101	110	111	...
2	12	20	102	112	120	200	1002	...
21	22	121	122	201	202	1021	1022	...
210	211	220	221	1210	1211	1220	1221	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$\uparrow$ Greedily Partitioned Sequence Base 3, Base 3/2 Construction $\downarrow$

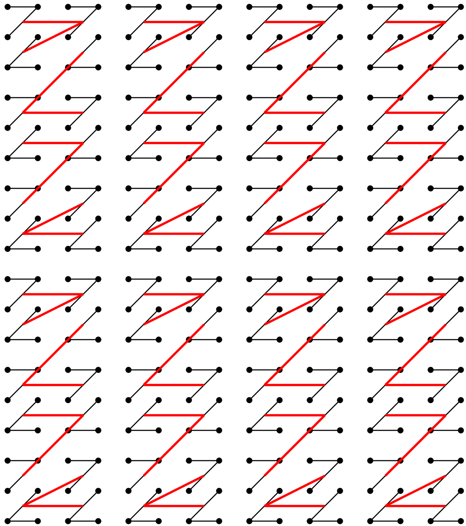
0	1	10	11	100	101	110	111	...
2			200	102	120	112	2000	...
21	22	201	202	121	122	2001	2002	...
210	211	220	221	2010	2011	2020	2021	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Properties on the Grid

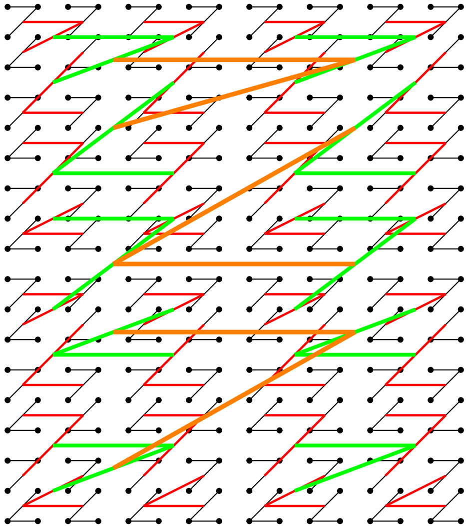
0	1	10	11	100	101	110	111	...
2	20	12	200	102	120	112	2000	...
21	22	201	202	121	122	2001	2002	...
210	211	220	221	2010	2011	2020	2021	...
212	211	222	2110	2012	2200	2022	2210	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱



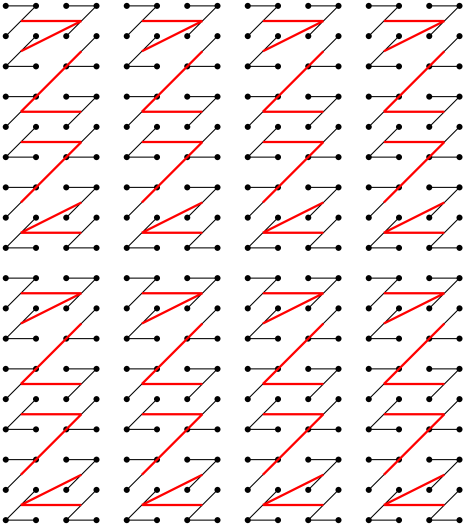
Properties on the Grid (cont.)



Final Connections



Zooming-out Procedure



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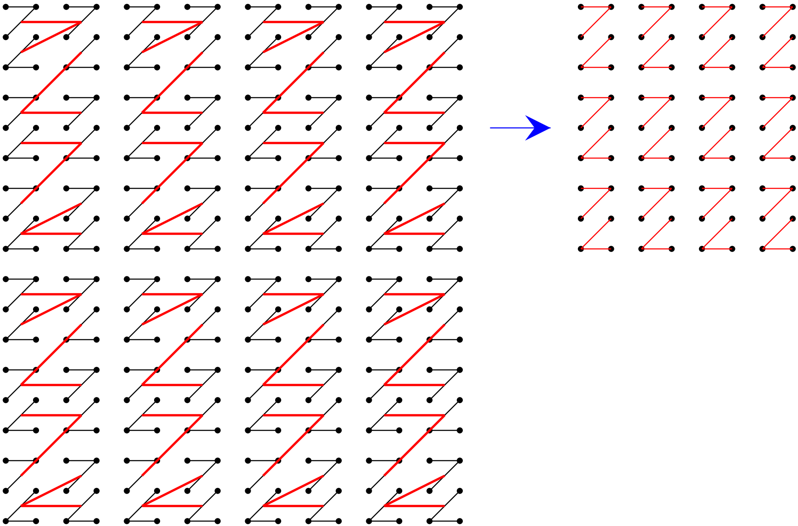
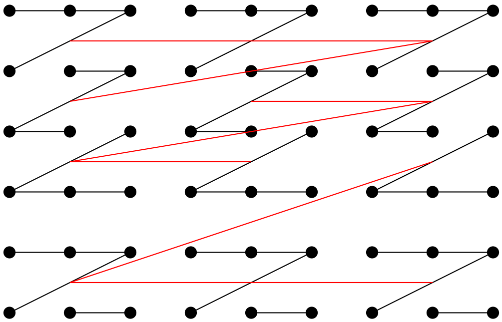


Future

Conjecture: For prime p, "integers" in base $\frac{p}{p-1}$ can be organized in a grid (equivalently constructed to the base 3/2 grid) such that each row is a greedily partitioned p-free sequence in base p.



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Some References

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