Quotients of Tropical Moduli Spaces

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Algebraic Geometry

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We care about compactifications of varieties; a space is compact if limits exist. In our example, we want to add in $z \to \infty$.

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Consider some variety V in $\mathbb{C}^2$. We map elements of $V \cap (\mathbb{C}^\times)^2$ to $\mathbb{R}^2$ by:

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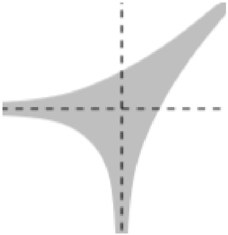
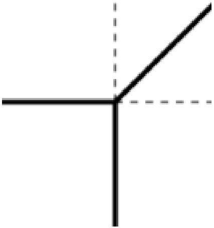


Figure: Amoeba of $x + y = 1$ (from [G])

Tropicalization, Continued

By taking $t \to \infty$,



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For a polynomial

$$f(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j,$$

its tropicalization is the tropical polynomial

$$\operatorname{trop} f(x, y) = g(x, y) = \bigoplus (c_{i,j} \odot ix \odot jy).$$

where

$$c_{i,j} = \lim_{t \to \infty} |a_{i,j}| = \begin{cases} -\infty, & a_{i,j} = 0 \\ 0, & a_{i,j} \neq 0 \end{cases}.$$

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Theorem (Kapranov 2000)

$$\mathbb{V}(\operatorname{trop} f(x_1, \ldots, x_n)) = \operatorname{trop} \mathbb{V}(f(x_1, \ldots, x_n)).$$

Application: Approximating Roots

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Compute the magnitudes of the roots of

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Actual roots: $x \approx -10^{14}$, $-5000 \pm 10^8 i$.

Moduli Spaces

Automorphisms on $\mathbb{P}^1$, are Möbius transformations:

$$z \to \frac{az+b}{cz+d}$$

i.e. maps that preserve cross ratio:

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}.$$

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We are interested in studying the moduli space $\mathcal{M}_{0,n}$ of n distinct points on the projective line $\mathbb{P}^1$ up to automorphism.

More on Moduli Spaces

For $n \geq 3$, we can send $P_1 \to 0, P_2 \to \infty$, $P_3 \to 1$, and $P_{3+n} \to (P_1, P_2; P_{3+n}, P_3)$.

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For $n \geq 3$, $\mathcal{M}_{0,n}$ is the configuration space of $n-3$ points in $\mathbb{C}^\times - \{1\}$.

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For a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ with $\vec{x} \cdot \vec{1} = 0$, consider the space $\mathcal{M}(\vec{x})$ of rational functions on $\mathbb{P}^1$ up to automorphism whose zeroes have order $x_1, \ldots, x_n$.

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Understanding the behavior of $\mathcal{M}_{0,n}$ in $\mathcal{M}(\vec{x})$ tells us about $\mathcal{M}_{0,n}$.

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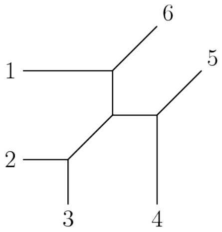

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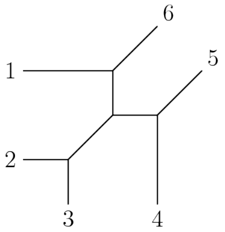
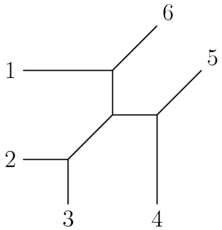


Figure: This tropical conic is a member of $\mathcal{M}_{0,6}^{\text{trop}}$.

The tree structure (with lengths ignored) is the combinatorial type.

$\mathcal{M}_{0,n}^{\text{trop}}$ as a fan



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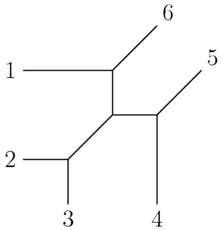

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This gives a fan structure: faces of cones are also cones.

Figure: A polyhedral fan in $\mathbb{R}^2$ (from [MS])

$\mathcal{M}_{0,n}^{\text{trop}}$ as a fan

Theorem (Speyer and Sturmfels 2006)

$\mathcal{M}_{0,n}^{\text{trop}}$ can be embedded as a tropical fan in $\mathbb{R}^{\binom{n}{2}-n}$.

$\mathcal{M}(\vec{x})^{\text{trop}}$

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The set of such maps is $\mathcal{M}(\vec{x})^{\text{trop}}$.

Example of the Balancing Condition

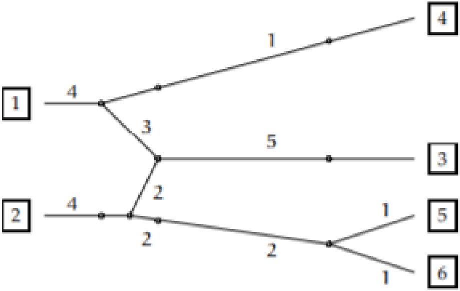

Figure: This is an element of $\mathcal{M}(\langle 4, 4, -5, -1, -1, -1 \rangle)$ (from [CMR])

Understanding $\mathcal{M}(\vec{x})^{\text{trop}}$

Proposition (Well known)

$$\mathcal{M}(\vec{x})^{\text{trop}} = \mathcal{M}_{0,n}^{\text{trop}} \times \mathbb{R}.$$

In particular, for fixed $\vec{x}$ and element T of $\mathcal{M}_{0,n}^{\text{trop}}$, the corresponding element of $\mathcal{M}(\vec{x})^{\text{trop}}$ is fixed up to shifting.

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In fact,

Theorem (Tevelev)

The tropicalization trop $\mathcal{M}(\vec{x})$ *of the variety* $\mathcal{M}(\vec{x})$ *is* $\mathcal{M}(\vec{x})^{\text{trop}}$.

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Goal: Canonical fan structure on $\mathcal{M}(\vec{x})^{\text{trop}} = \mathcal{M}_{0,n}^{\text{trop}} \times \mathbb{R}$ without cones containing lines through the origin.

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Two ways to do this:

1. Divide into cones based on which (interior) vertices are mapped to $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$; "sign subdivision".
2. Divide into cones based on the relative order of 0 and the interior vertices; "order subdivision".

Current Work

Goal: Understand the embedding of $\mathcal{M}_{0,n}^{\text{trop}}$ in $\mathcal{M}(\vec{x})^{\text{trop}}$, or the (Chow) quotient map $\mathcal{M}(\vec{x})^{\text{trop}} \to \mathcal{M}_{0,n}^{\text{trop}}$.

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Theorem (Own)

The "universal family" of the sign subdivision of $\mathcal{M}(\vec{x})^{\text{trop}}$ to make the quotient map a fan morphism is in fact the order subdivision.

Future Work and Difficulties

Goal: Extend to positive genera.

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Difficulty: $\mathcal{M}_{g,n}^{\text{trop}} \times \mathbb{R}^k \neq \mathcal{M}(\vec{x})^{\text{trop}}$ for any k.

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