The Shuffle Algebra of the Hilbert Scheme of Points of the Plane

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Algebras

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A **module** M over a ring R is a set of elements that can be added together and multiplied by a scalar $\lambda \in R$. An **algebra** is a module equipped with a product between elements in M that outputs another element in M.

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The set of $n \times n$ square matrices in R

The Shuffle Algebra

For a ring R, the shuffle algebra A^R is the subset of the set of symmetric rational functions in arbitrarily many variables with coefficients in R, generated by 1 variable functions.

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$$F(a,b)*G(c,d) = F(a,b)G(c,d)+F(a,c)G(b,d)+F(a,d)G(b,c)$$

$$+F(b,c)G(a,d)+F(b,d)G(a,c)+F(c,d)G(a,b).$$

The Integral Shuffle Algebra

Definition

The **integral shuffle algebra** is a subset of $\bigoplus_{k\geq 0} \mathrm{Sym}_{\mathbf{R}}(z_1, ..., z_k)$ and is the shuffle algebra over the ring $\mathbf{R} = \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}]$.

The shuffle product is

$$P(z_1, \ldots, z_k) * Q(z_1, \ldots, z_l) =$$

$$\frac{1}{k!l!} \sum_{\mathrm{sym}} P(z_1, \ldots, z_k) Q(z_{k+1}, \ldots, z_{k+l}) \prod_{\substack{1\leq i\leq k \\ k<j\leq k+l}} \frac{(z_i - q_1 q_2 z_j)(z_j - q_1 z_i)(z_j - q_2 z_i)}{z_i - z_j}.$$

We want to find conditions to determine whether a given symmetric rational function is in the integral shuffle algebra.

The Fractional Shuffle Algebra

The **fractional shuffle algebra** is the shuffle algebra over the ring $\mathbf{K} = \mathbb{C}(q_1, q_2)$ with the same shuffle product as the integral shuffle algebra.

Theorem (Negut, 2014)

A symmetric rational function $p(z_1, \ldots, z_k)$ is in the fractional shuffle algebra if and only if it is a Laurent polynomial ($p \in \mathbf{K}[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$) and it satisfies the **wheel conditions***:*

$$p(z_1, q_1 z_1, q_1 q_2 z_1, z_4, z_5, \ldots, z_k) = p(z_1, q_2 z_1, q_1 q_2 z_1, z_4, z_5, \ldots, z_k) = 0.$$

These conditions are necessary but not sufficient for the integral shuffle algebra.

Ideals

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An **ideal** of a ring R is a subset of R that is closed under addition and multiplication by elements of R. An ideal can be written as $(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are the generators of the ideal.

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Ideal form of wheel conditions: $p(z_1, q_1 z_1, q_1 q_2 z_1, \ldots) = 0$ if and only if $p \in (q_1 q_2 z_1 - z_3, q_1 z_1 - z_2, q_2 z_2 - z_3)$ of the ring of Laurent polynomials.

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Ideals can also be thought of as R-modules that are contained in R.

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A quotient R/I of a ring R by an ideal I is the ring of equivalence classes in the ring where two elements a and b are equivalent if $a - b \in I$.

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$R[x]/(x) = R$

$R[x]/(x^2) = \{ax + b \mid a, b \in R\}$

The Hilbert Scheme of Points in the Plane

The Hilbert scheme Hilb_n of n points in the plane is the set of ideals $I \subset \mathbb{C}[x, y]$ such that the dimension of $\mathbb{C}[x, y]/I$ as a vector space over $\mathbb{C}$ is n.
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$$\mathbb{C}[x, y]/(x) = \{a + by + cy^2 + \dots\} \text{ so } (x) \notin \text{Hilb}_n \text{ for any } n$$

Relation of Shuffle Algebra to Hilbert Scheme

Theorem (Schiffmann and Vasserot, 2013)

Consider the equivariant K-theory group $K^T(Hilb_n)$ of the Hilbert scheme and let

$$L_{\mathbf{R}} = \bigoplus_{n \geq 0} K^T(Hilb_n), \quad L_{\mathbf{K}} = L_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{K}$$

where $\mathbf{R} = \mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}]$ and $\mathbf{K} = \mathbb{C}(q_1, q_2)$.
Then $L_{\mathbf{R}}$ is a module over the integral shuffle algebra and $L_{\mathbf{K}}$ is a module over the fractional shuffle algebra.

Ongoing Work

Theorem

Let $A_k^{\mathbf{R}}$ be the subset of the integral shuffle algebra consisting of functions in k variables. Then the following hold:

- $A_k^{\mathbf{R}}$ *is an ideal of* $\mathbf{R}[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$ *for all* k.
- $A_2^{\mathbf{R}}$ *is the ideal* $(z_1 * z_1^0, z_1^0 * z_1^0)$ *of* $\mathbf{R}[z_1^{\pm 1}, z_2^{\pm 1}]$.
- *As an ideal of* $\mathbf{R}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, $A_3^{\mathbf{R}}$ *is generated by the elements* $z_1^{d_1} * z_1^{d_2} * z_1^0$ *for* $0 \leq d_1 \leq 2$, $0 \leq d_2 \leq 1$.

Ongoing Work

Recall the ideal form of the wheel conditions:

$$p \in (q_1 q_2 z_1 - z_3, q_1 z_1 - z_2, q_2 z_2 - z_3),$$

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We create a similar condition from the generators of $A_2^{\mathbf{R}}$:

Theorem

$A_k^{\mathbf{R}}$ is contained in the ideal

$$(z_1 * z_1^0, z_1^0 * z_1^0)$$

of $\mathbf{R}[z_1^{\pm 1}, \ldots, z_k^{\pm 1}]$ for $k \geq 2$.

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- Find a computer algebra system/algorithm to calculate ideals of $\mathbf{R}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, as hand calculations are not feasible:

$$P(z_1, \ldots, z_k) * Q(z_1, \ldots, z_l) =$$

$$\frac{1}{k!l!} \sum_{\mathrm{sym}} P(z_1, \ldots, z_k) Q(z_{k+1}, \ldots, z_{k+l}) \prod_{\substack{1 \le i \le k \\ k < j \le k+l}} \frac{(z_i - q_1 q_2 z_j)(z_j - q_1 z_i)(z_j - q_2 z_i)}{z_i - z_j}.$$

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- Try to prove that conditions are sufficient or find new ways to generate conditions that can be proven.

Difficulties of the Project: $z_1^0 * z_1^0 * z_1^0$

$$z_1^0 * z_1^0 * z_1^0 = 6q_1^3q_2^3z_1^4z_2^2 + (-3q_1^2q_2^2 - 3q_1^3q_2^2 - 3q_1^2q_2^3 + 6q_1^3q_2^3 - 3q_1^4q_2^3 - 3q_1^3q_2^4 - 3q_1^4q_2^4)z_1^3z_2^3 + 6q_1^3q_2^3z_1^2z_2^4$$
$$+(-3q_1^2q_2^2 - 3q_1^3q_2^2 - 3q_1^2q_2^3 + 6q_1^3q_2^3 - 3q_1^4q_2^3 - 3q_1^3q_2^4 - 3q_1^4q_2^4)z_1^4z_2z_3 + (q_1q_2 + 4q_1^2q_2 + q_1^3q_2 + 4q_1q_2^2 - 7q_1^2q_2^2$$
$$-7q_1^3q_2^2+4q_1^4q_2^2+q_1q_2^3-7q_1^2q_2^3+24q_1^3q_2^3-7q_1^4q_2^3+q_1^5q_2^3+4q_1^2q_2^4-7q_1^3q_2^4-7q_1^4q_2^4+4q_1^5q_2^4+q_1^3q_2^5+4q_1^4q_2^5+q_1^5q_2^5)z_1^3z_2^2z_3$$
$$+(q_1q_2+4q_1^2q_2+q_1^3q_2+4q_1q_2^2-7q_1^2q_2^2-7q_1^3q_2^2+4q_1^4q_2^2+q_1q_2^3-7q_1^2q_2^3+24q_1^3q_2^3-7q_1^4q_2^3+q_1^5q_2^3+4q_1^2q_2^4-7q_1^3q_2^4$$
$$-7q_1^4q_2^4+4q_1^5q_2^4+q_1^3q_2^5+4q_1^4q_2^5+q_1^5q_2^5)z_1^2z_2^3z_3+(-3q_1^2q_2^2-3q_1^3q_2^2-3q_1^2q_2^3+6q_1^3q_2^3-3q_1^4q_2^3-3q_1^3q_2^4-3q_1^4q_2^4)z_1z_2^4z_3$$
$$+6q_1^3q_2^3z_1^4z_3^2+(q_1q_2+4q_1^2q_2+q_1^3q_2+4q_1q_2^2-7q_1^2q_2^2-7q_1^3q_2^2+4q_1^4q_2^2+q_1q_2^3-7q_1^2q_2^3+24q_1^3q_2^3-7q_1^4q_2^3+q_1^5q_2^3+4q_1^2q_2^4$$
$$-7q_1^3q_2^4-7q_1^4q_2^4+4q_1^5q_2^4+q_1^3q_2^5+4q_1^4q_2^5+q_1^5q_2^5)z_1^3z_2z_3^2+(-1-2q_1-2q_1^2-q_1^3-2q_2-q_1q_2+6q_1^2q_2-q_1^3q_2-2q_1^4q_2-2q_2^2$$
$$+6q_1q_2^2-13q_1^2q_2^2-13q_1^3q_2^2+6q_1^4q_2^2-2q_1^5q_2^2-q_2^3-q_1q_2^3-13q_1^2q_2^3+42q_1^3q_2^3-13q_1^4q_2^3-q_1^5q_2^3-q_1^6q_2^3-2q_1q_2^4+6q_1^2q_2^4$$
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$$+(q_1q_2+4q_1^2q_2+q_1^3q_2+4q_1q_2^2-7q_1^2q_2^2-7q_1^3q_2^2+4q_1^4q_2^2+q_1q_2^3-7q_1^2q_2^3+24q_1^3q_2^3-7q_1^4q_2^3+q_1^5q_2^3+4q_1^2q_2^4-7q_1^3q_2^4-7q_1^4q_2^4$$
$$+4q_1^5q_2^4+q_1^3q_2^5+4q_1^4q_2^5+q_1^5q_2^5)z_1z_2^3z_3^2+6q_1^3q_2^3z_2^4z_3^2+(-3q_1^2q_2^2-3q_1^3q_2^2-3q_1^2q_2^3+6q_1^3q_2^3-3q_1^4q_2^3-3q_1^3q_2^4-3q_1^4q_2^4)z_1^3z_3^3$$
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$$+4q_1^5q_2^4+q_1^3q_2^5+4q_1^4q_2^5+q_1^5q_2^5)z_1^2z_2z_3^3+(q_1q_2+4q_1^2q_2+q_1^3q_2+4q_1q_2^2-7q_1^2q_2^2-7q_1^3q_2^2+4q_1^4q_2^2+q_1q_2^3-7q_1^2q_2^3+24q_1^3q_2^3$$
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