3-symmetric Graphs

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Symmetric permutations

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Definition

Call a pair of terms (p_i, p_j) in a permutation p an **inversion** if $i > j$ and $p_i < p_j$.

Example

In the permutation $1, 3, 2$, the pair $(3, 2)$ forms an inversion.

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In the permutation $1, 3, 2$, the pair $(3, 2)$ forms an inversion.

Property: A random permutation should have about an equal number of inversions as non-inversions.

Definition

Call a permutation 2-symmetric if it has the same number of inversions as non-inversions.

Examples of 2-symmetric permutations

Example

$4, 1, 2, 3$ is 2-symmetric.

- $(4, 1)$,$(4, 2)$,$(4, 3)$ are inversions, while
- $(1, 2)$,$(1, 3)$,$(2, 3)$ are not.

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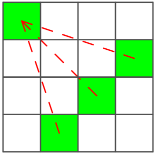

Figure: The permutation $4, 1, 2, 3$.

3-symmetric permutations

Definition

Call a permutation **3-symmetric** if a randomly chosen unordered triplet of points is equally likely to be ordered like each of the six permutations of $1, 2, 3$.

Example

The permutation $4, 7, 2, 9, 5, 1, 8, 3, 6$ is 3-symmetric (PRIMES 2018, Eric Zhang and Tanya Khovanova).

Generalize to k-symmetric easily.

Analogous definition for graphs

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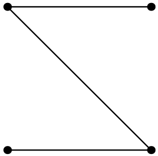

Figure: A 2-symmetric graph with 4 vertices.

Size restrictions

Assume G is 2-symmetric and G has n vertices.

Then since G must have $\frac{\binom{n}{2}}{2}$ edges, we need $n \equiv 0, 1 \mod 4$.

k-symmetric graphs

Extend the definition to k-symmetric graphs:

Definition

A graph G is k-**symmetric** if for any subgraph H of G with $|H| = k$, the density of H in G is the same as the probability that a randomly chosen graph on k vertices is isomorphic with H.

Note the analogy with k-symmetric permutations.

Example: 3-symmetric graphs

Consider $k = 3$. Then

- $\frac{1}{8}$ of triplets of points in G must be triangles,
- $\frac{3}{8}$ are paths of length 2,
- $\frac{3}{8}$ are single edges, and
- $\frac{1}{8}$ are independent sets.

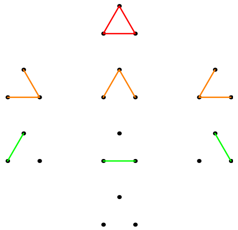

Figure: Possible graphs on 3 vertices.

Size restrictions

k-symmetric graphs restrict size as well. For $k = 3$, then $8 \mid \binom{n}{3}$ which implies $|G| \equiv 0, 1, 2, 8, 10 \mod 16$.

Theorem

If a graph G is k-symmetric, it is also m-symmetric for any $m < k$.

Sketch: Double counting subgraphs.

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Corollary

If G is 3-symmetric, then $|G| \equiv 0, 1, 8 \mod 16$.

Examples of 3-symmetric graphs

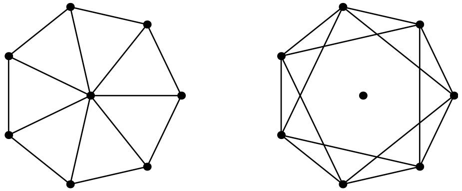

Figure: A wheel and its complement.

There are 74 graphs on 8 vertices that are 3-symmetric (verified by Prof. David Perkinson with a computer).

Inflation

A possible mechanism for generating 3-symmetric graphs.

Definition

For graphs G and H, define the **inflation** (or the **lexicographic product**) of G with respect to H as the graph inflate(G, H) with $|G||H|$ vertices where:

- Each vertex in G is replaced with a graph isomorphic to H, and
- If H_i, H_j are the graphs that correspond to nodes i and j in G, and $\{i, j\}$ is an edge in G, then an edge is drawn between each vertex in H_i to each vertex in H_j.

Inflation Example

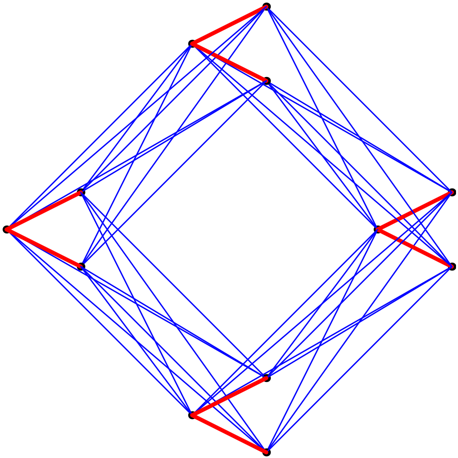

Figure: inflate(□, ⋀)

Inflation preserves 3-symmetric graphs in the limit case.

Theorem

Let $G_1, G_2, \ldots$ be a sequence of 3-symmetric graphs whose sizes go to ∞, and H also be 3-symmetric. Then the densities of any subgraph of size 3 in the inflation of H into G_i will tend to their expected probabilities in a random graph.

3-symmetric graph of size 16

- We used a C++ program to add edges randomly between two wheels.
- Randomness: Enumerate the 64 “cross-edges”, generate a random permutation of length 64, and take the first 32 and use them as the edges.
- This creates a 2-symmetric graph; check whether it is 3-symmetric

3-symmetric graph of size 16

The number of 2-symmetric graphs generated this way that were also 3-symmetric was 561 out of 10^5 trials ($\approx 0.56\%$).

A 3-symmetric graph of size 16

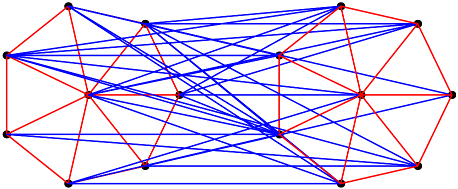

Figure: A 3-symmetric graph of size 16 formed by connecting two wheels.

Statistics of 3-symmetric graphs of size 16

Maximum clique sizes (500 trials):

Max Clique	Frequency
4	41
5	436
6	23

Max degree:

Max Degree	Frequency
9	1
10	115
11	260
12	109
13	14
14	1

Attempting to construct k-symmetric graphs, $k > 3$

By the size restrictions, 4-symmetric graphs have at least 256 vertices.

Computational limits:

- $\binom{256}{4}$ subgraphs to consider
- Need to solve graph isomorphism problem for larger k

Future

- Conjecture: 3-symmetric graphs with $8n$ vertices exist for all $n \geq 1$.
- If true, find asymptotics on the number of 3-symmetric graphs
- Find mechanisms to generate k-symmetric graphs

Acknowledgements

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- My parents