Cutting and Gluing Surfaces

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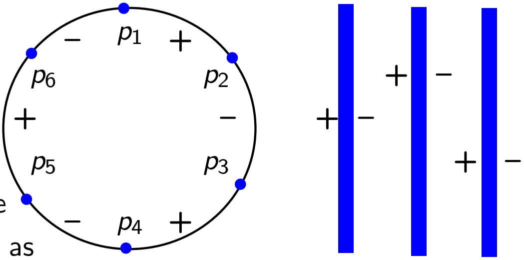
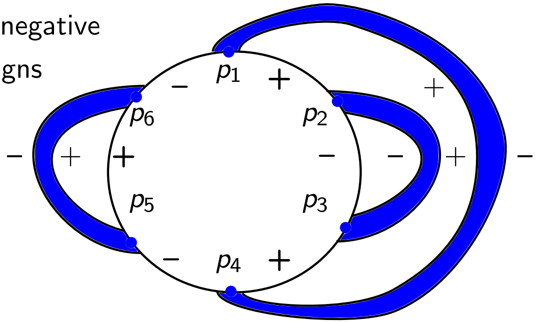
Setup

- A disk with 6 vertices
- Assign +/- to arcs alternately
- Attach 3 strips to the disk along vertices
- Each strip has a positive side and a negative side as shown



Boundaries

- The strips connect p_1 to p_4, p_2 to p_3 and p_5 to p_6
- This gives us positive and negative boundaries based on the signs of the arcs and strips
- The new surface has 4 boundary components.
- Two positive boundaries
- Two negative boundaries



General case

- Let D be a disk, with $2n$ vertices $p_1,\ldots,\ p_{2n}$ on ∂D.
- Here n is odd.
- Assign $+/-$ to arcs between adjacent vertices
- Attach n strips along vertices according to a pairing $\mathcal{P}$. Call the resulting surface $D_{\mathcal{P}}$.

Definition

A **pairing** $\mathcal{P}$ is a collection of n pairs

$$\mathcal{P} = \{(i_1, j_1), \ldots, (i_n, j_n)\},$$

so that
(1). $i_k \not\equiv j_k \pmod 2$ which ensures that the surface is orientable
(2). $\{i_1, j_1, \ldots, i_n, j_n\} = \{1, \ldots, 2n\}$.

Signature

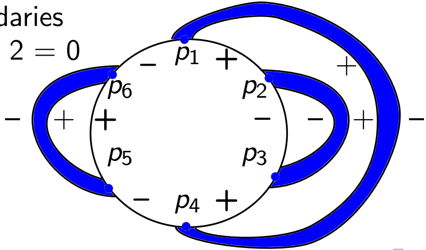
Definition

The **signature** of $D_{\mathcal{P}}$ is the difference between the number of positive boundaries and the number of negative boundaries.

Definition

A pairing $\mathcal{P}$ is **balanced** if its signature is 0.

- Two positive boundaries
- Two negative boundaries
- The signature is 2 - 2 $= 0$



Connected Pairings

Definition

A **valid cut and glue operation** switches a pairing from including (e_1, o_1) and (e_2, o_2) to (e_1, o_2) and (e_2, o_1) whenever e_1, e_2, o_1, o_2 are in 4 separate boundary components.

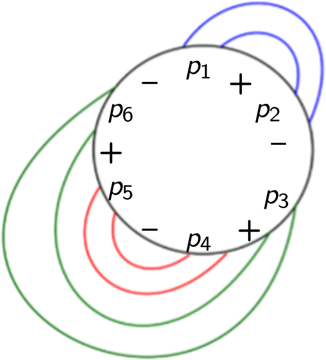
Example

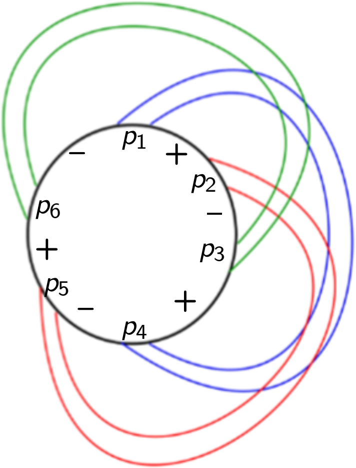
A valid cut and glue operation on the pairing $\{\mathbf{(1,\ 2)},\ \mathbf{(4,\ 5)},\ (3,\ 6)\}$ could change it to $\{\mathbf{(1,\ 4)},\ \mathbf{(2,\ 5)},\ (3,\ 6)\}$.

Definition

Two pairings are **connected** if you can reach one from the other through legal cut and glue operations.

Example in Two Dimensions



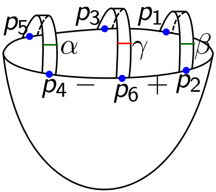


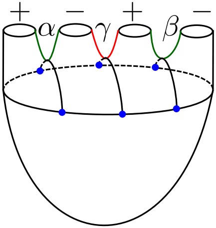
Note how 1,2,4,5 are in 4 different boundaries.

Examples of Cut and Glue in Three Dimensions

Now we clarify what we mean by the vertices occupying four different boundary components.

- Cut and paste along $(4, 5)$ and $(1, 2)$ is valid. This corresponds to arcs α and β.
- Cut and paste along $(4, 5)$ and $(3, 6)$ is invalid. This corresponds to arcs α and γ.

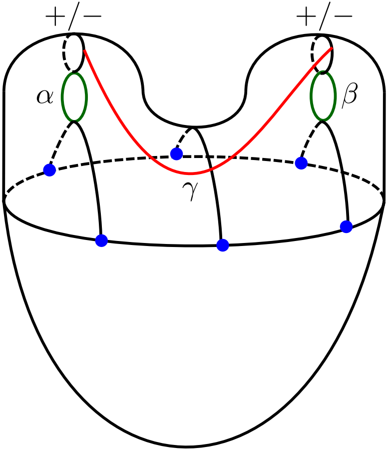




Two topologically equivalent surfaces related by a deformation.

Cut and Glue in Three Dimensions cont.

- After gluing $+$ boundaries of $D_{\mathcal{P}}$ to $-$ boundaries of $D_{\mathcal{P}}$, $D_{\mathcal{P}}$ becomes a closed surface.
- The curves α and β becomes closed curves while γ remains an arc.
- There are no ways to make α and γ both closed: cut and glue along α and γ is not allowed. Neither are β and γ.



Examples of pairings with different signatures

Here, adjacent integers represent ones connected with strips, while A denotes 10 as in hexadecimal.

The signature for 14.25.36.7A.89 is 2 - 2 = 0
The signature for 14.25.38.7A.69 is 1 - 1 = 0
The signature for 16.27.38.49.5A is 1 - 1 = 0
The signature for 12.34.56 is 3 - 1 = 2

Proposition

Any two connected pairings have the same signature.

Results, Conjectures and Extensions

Main Theorem (Kavi, Li)

All balanced pairings on a surface with one boundary are connected.

Conjecture

Two pairings are connected if and only if they have the same signature.

Extension

What about surfaces with more than one boundary?

General Surfaces

Replace D by a general, oriented surface S.

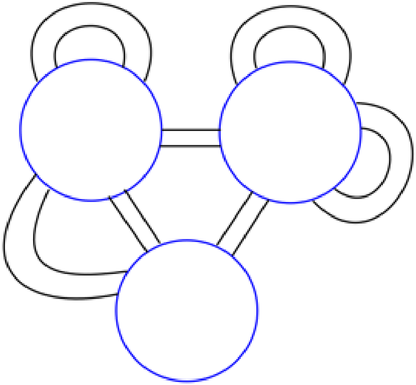


Figure: Surface with more than one circle

Balanced Generalized Pairings

Definition

The **signature** of a pairing $\mathcal{P}$ is the difference between the number of positive boundaries and the number of negative boundaries.

Definition

A pairing $\mathcal{P}$ is **balanced** if its signature is 0.

Future Research

Conjecture

All balanced pairings on a general surface are connected.

Conjecture

Two pairings are connected if and only if they have the same signature.

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