Fractals

Hausdorff Dimension, the Koch Curve, and Visibility

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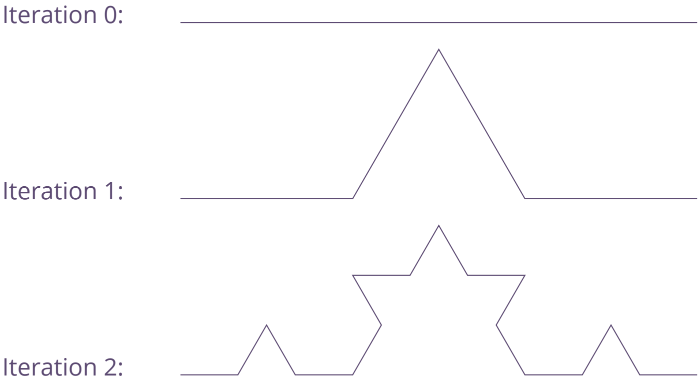
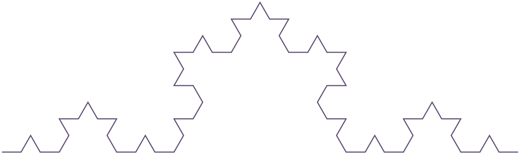
Koch Curve

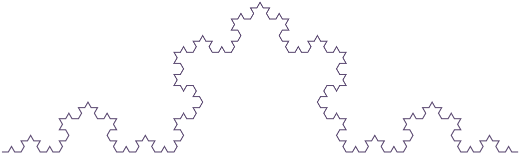


Koch Curve

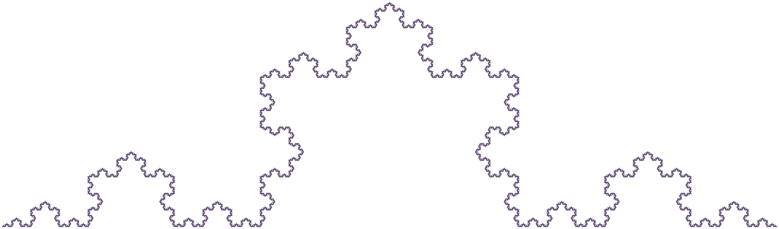
Iteration 3:



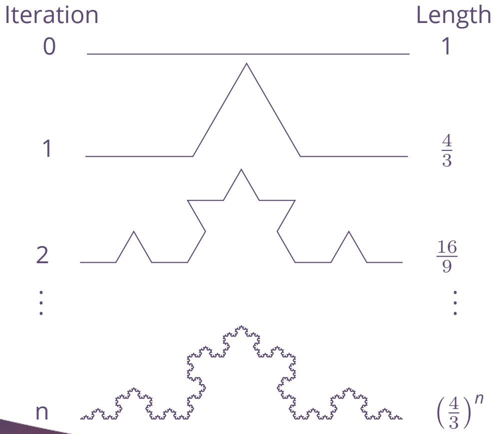
Iteration 4:



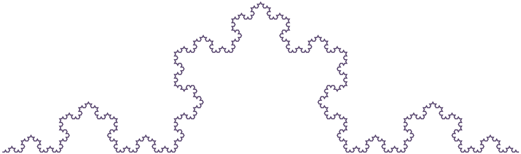
Koch Curve



Length of the Koch Curve

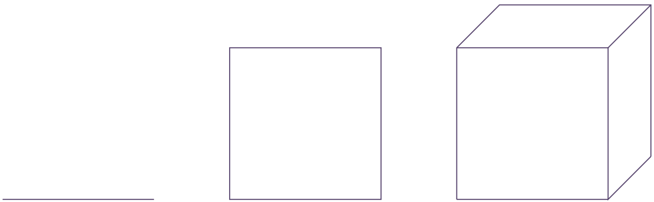


Length of the Koch Curve

n  $\left(\frac{4}{3}\right)^n$

$$\lim_{n\to\infty} \text{length}(K) = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty$$

The Scaling Property



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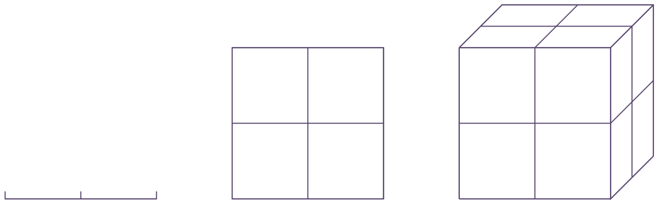

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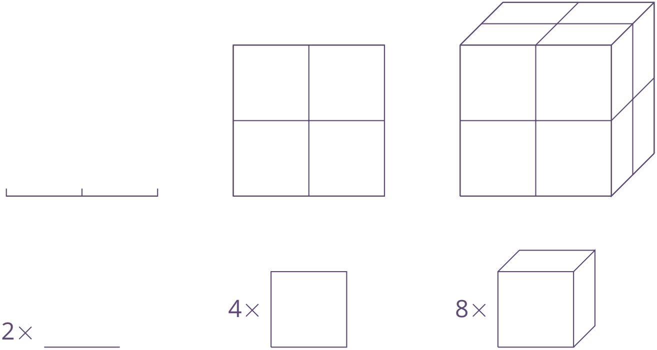
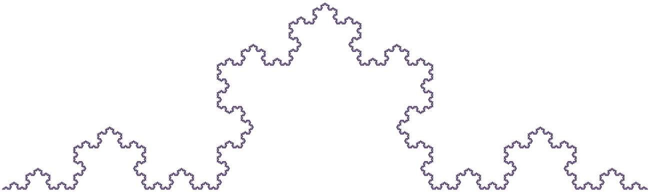


The Scaling Property

set X	scaling factor c	# of pieces m	dimension $\dim$
line	1/2	2	1
square	1/2	4	2
cube	1/2	8	3

$$\dim_{\mathrm{H}} X = \log_{c^{-1}} m$$

Dimension of the Koch Curve



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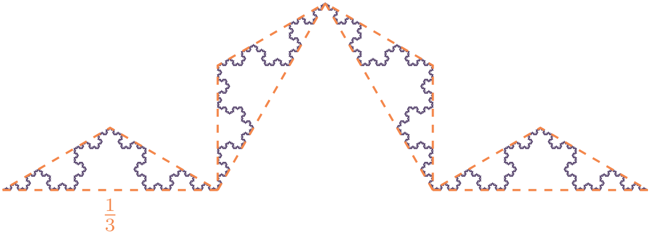
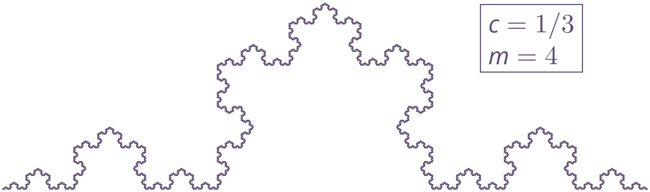
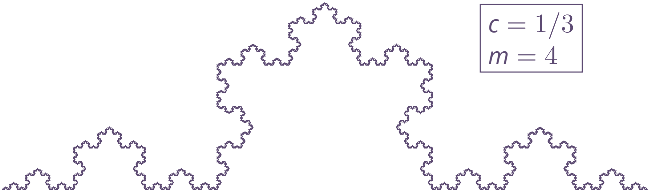


Dimension of the Koch Curve



$c = 1/3$
$m = 4$

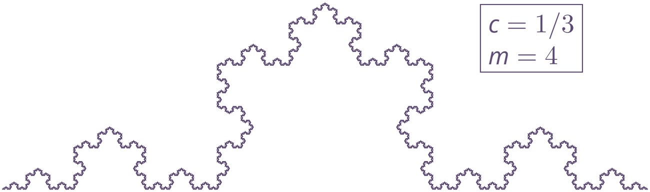
Dimension of the Koch Curve



$c = 1/3$
$m = 4$

$$\dim_{\mathrm{H}} K = \log_{c^{-1}} m = \log_3 4$$

Dimension of the Koch Curve

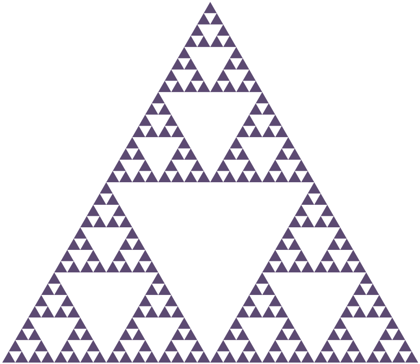


$c = 1/3$
$m = 4$

$$\dim_{\mathrm{H}} K = \log_{c^{-1}} m = \log_3 4$$

$$1 < \dim_{\mathrm{H}} K = 1.262\cdots < 2$$

Dimension of the Sierpinski Gasket



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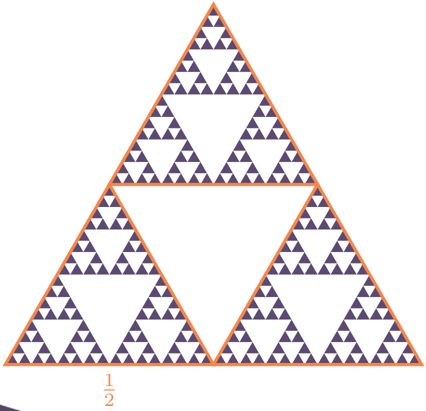

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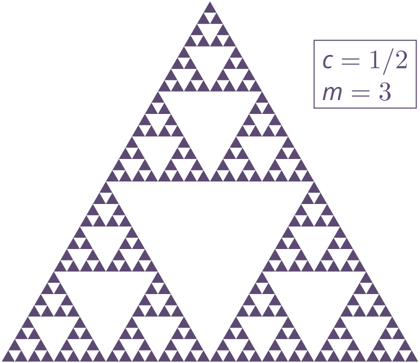

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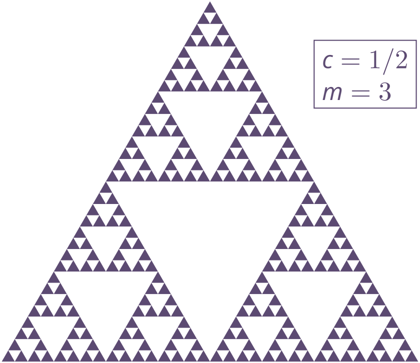
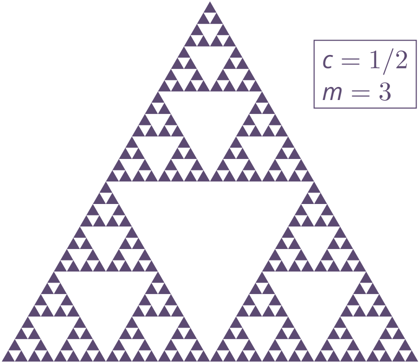
Dimension of the Sierpinski Gasket



$c = 1/2$
$m = 3$

$$\dim_{\mathrm{H}} S = \log_{c^{-1}} m = \log_2 3$$

Dimension of the Sierpinski Gasket



$c = 1/2$
$m = 3$

$$\dim_{\mathrm{H}} S = \log_{c^{-1}} m = \log_2 3$$

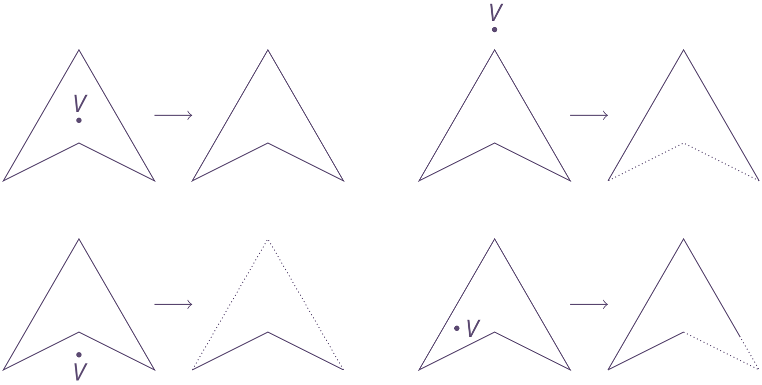
$$1 < \dim_{\mathrm{H}} S = 1.585 \cdots < 2$$

Visibility

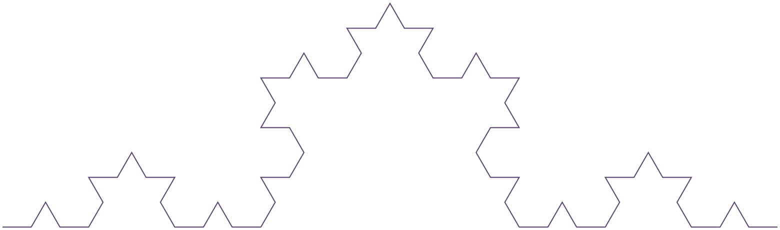
A point P in a set X is *visible* from a point V if there are no other points in X on the line segment connecting P and V.

The collection of all points in X visible from V is denoted X_V.

Visibility



Koch Curve Visible from $(0, \infty)$



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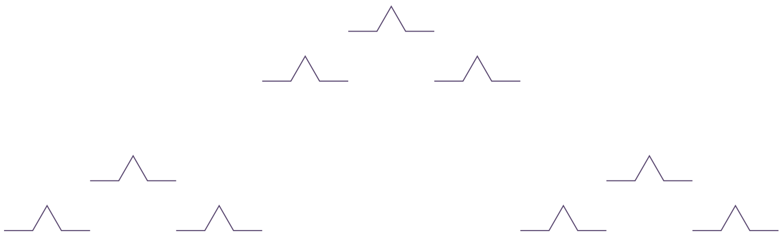

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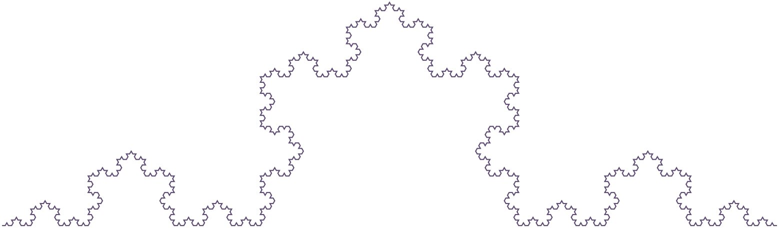

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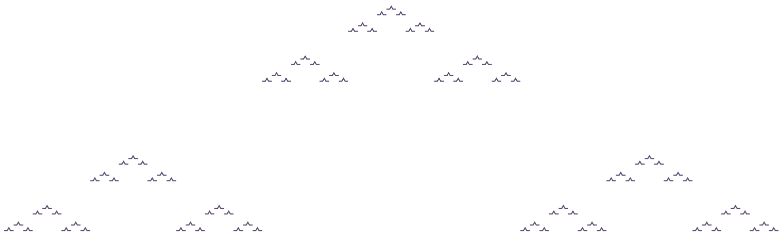

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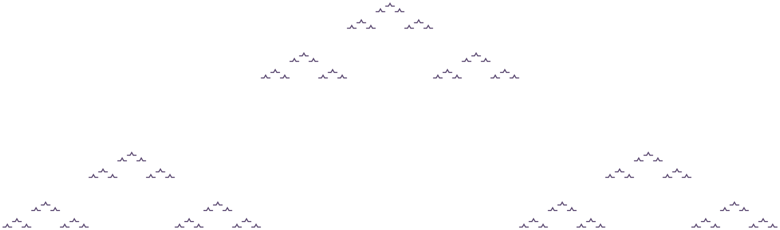
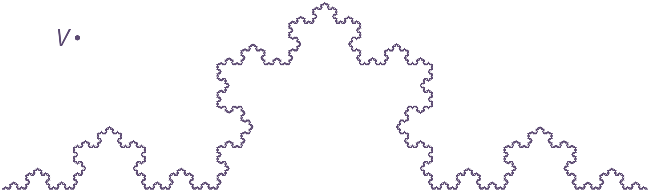


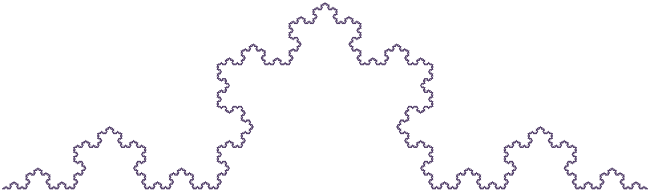
$$\dim_{\mathrm{H}} K_{(0,\infty)} = \log_3 3 = 1$$

Project Goal



$$\dim_{\mathrm{H}} K_V = ?$$

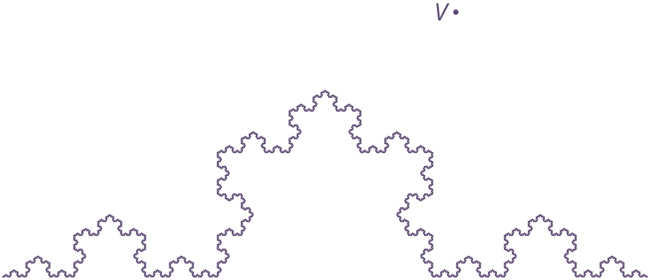
Project Goal



$V\bullet$

$$\dim_{\mathrm{H}} K_V = ?$$

Project Goal



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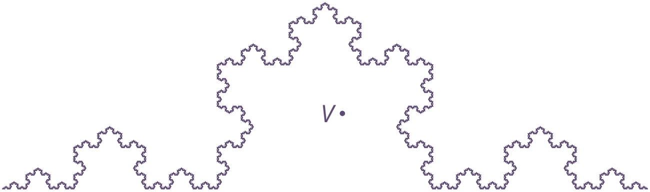


$$\dim_{\mathrm{H}} K_V = ?$$

Hausdorff Measure

The *s-dimensional Hausdorff measure* of a set $F \subset \mathbb{R}^n$ is defined to be

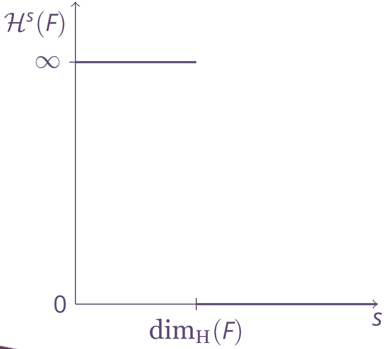
$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F),$$

where $\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$.

Hausdorff Dimension

The *Hausdorff dimension* $\dim_{\mathrm{H}} F$ of a set $F \in \mathbb{R}^n$ is defined to be

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$



Preliminary Results

- $\dim_{\mathrm{H}}(K_{V_\infty}) = 1$ when V_∞ is an arbitrary point at infinity.

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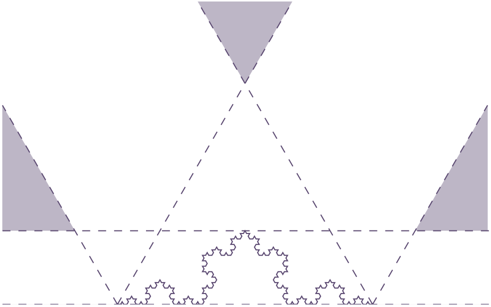

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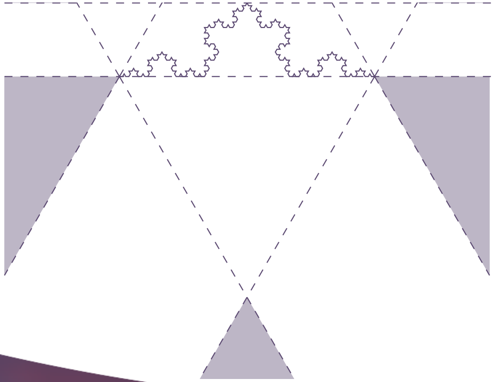

Future Research

- Calculate the Hausdorff dimension of K_V for any $V \in \mathbb{R}^2$.

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- Calculate the Hausdorff dimension for other fractals with visibility conditions.
- Generalize the results: for a fractal F, when is $\dim_{\mathrm{H}}(F_V) > 1$?

Acknowledgments

- PRIMES
- Dr. Tanya Khovanova, for mentoring this project
- Prof. Larry Guth, for suggesting the problem
- Friends and family

References

[1] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*. John Wiley & Sons, 2004.