PRIMES Math Problem Set

PRIMES 2018

Due December 1, 2017

Dear PRIMES applicant:

This is the PRIMES 2018 General Math Problem Set. Please send us your solutions as part of your PRIMES application by **December 1, 2017**. For complete rules, see `http://math.mit.edu/research/highschool/primes/program/apply.php`

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`smith-solutions`". Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General Math Problems

Problem G1. We have n fair six-sided dice, labeled 1 through 6. Let p_n be the probability that when rolled, the product of all n numbers shown is at most 6.

(a) Compute the value of p_2.

(b) Determine p_n for any integer $n \geq 2$.

Problem G2. Fix an integer $d \geq 1$, and consider polynomials $P(x)$ of degree d which satisfy

$$P(n) = n + \frac{1}{n}$$

for $n = 1, 2, \ldots, 99$. For each d, determine all possible values of $P(100)$ or show that no such polynomials P of degree d exist.

Problem G3. There are 100 marbles in a bag: 30 red marbles, 60 green marbles and 10 yellow marbles. We select three of them uniformly at random, independently and with replacement (meaning we put the ball back after it is removed). Let E be the event "all three marbles are the same color".

(a) Find the probability of E.

(b) Find the probability of E, given that at least one selected marble is red.

(c) Find the probability of E, given that the selected marbles aren't all different colors.

Problem G4. A pair (σ, τ) of permutations on $\{0, 1, \ldots, n-1\}$ is *balanced* if the following map is also a bijection on $\{0, 1, \ldots, n-1\}$:

$$x \mapsto (\sigma(x) + \tau(x)) \mod n.$$

(Here, $a \mod n$ means the remainder when a is divided by n.)

(a) Does there exist a balanced pair when $n = 3$? (Either give an example or prove none exists.)

(b) Does there exist a balanced pair when $n = 4$? (Either give an example or prove none exists.)

(c) For which n does there exist a balanced pair?

Problem G5. Consider the following six points in the coordinate plane:

$$A = (0,1), \quad B = (0,3), \quad C = (1,4), \quad D = (4,9), \quad E = (6,7), \quad F = (6,8).$$

For a point P in the coordinate plane let $S(P) = PA + PB + PC + PD + PE + PF$.

(a) Prove that $S(P)$ is minimized at some point P.

(b) Determine the value of that minimum.

Problem G6. Let a, b, c be positive real numbers for which $\min(ab, bc, ca) \geq 1$.

(a) Prove that $\log(abc) \geq \sqrt[3]{(\log a)^3 + (\log b)^3 + (\log c)^3}$.

(b) Determine for which triples (a, b, c) the equality holds.

Problem G7. For each integer $n \geq 1$ let $\mathcal{T}_n$ denote the set of nondegenerate triangles whose side lengths are in $\{1, \ldots, n\}$. Moreover, for each triangle $\triangle ABC$, let

$$D(\triangle ABC) = \min\left(|AB - AC|, |BC - BA|, |CA - CB|\right).$$

(a) For each integer $n \geq 3$, determine the largest possible value of $D(\triangle ABC)$ over all triangles in $\mathcal{T}_n$.

(b) For which n is this maximum value achieved for a unique triangle in $\mathcal{T}_n$ (up to congruence)?

Advanced Math Problems

Problem M1. Let P be a partially ordered set (poset) with 12 elements. Given that P has width 2, what is the maximum number of linear extensions that P can have?

(A *linear extension* of a poset P is a total ordering of the elements compatible with the partial order. The *width* of a partially ordered set is the largest size of a subset in which no two elements are comparable; this is the size of the largest antichain. See `https://en.wikipedia.org/wiki/Partially_ordered_set` for the definition of a poset.)

Problem M2. Consider the infinite series

$$S = \sum_{n=2}^{\infty} \left[\log(n^3 + k) - \log(n^3 - k)\right].$$

where $k \in (0, 8)$ is a real number. (Here log denotes the natural logarithm.)

(a) Prove that S converges for any k.

(b) For $k = 1$, compute S.

Problem M3. Consider integrable functions $f : [0, \pi] \to [-1, 1]$ such that we have $\int_0^\pi f(x)\, dx = 0$, and let

$$S(f) = \int_0^\pi f(x) \sin x \, dx.$$

Find a constant M, as small as you can, for which $|S(f)| \leq M$.

Problem M4. For integers $n \geq 0$ let

$$a_n = \sum_{i=0}^{n} \frac{1}{i+1} \binom{n+i}{n-i} \binom{2i}{i}.$$

Identify the sequence $(a_n)_n$ by name and prove that a_n is the claimed sequence. (You may use the Online Encyclopedia of Integer Sequences, `http://oeis.org/`.)

Problem M5. Let G be a group with presentation given by

$$G = \left\langle a, b, c \mid ab = c^2 a^4,\ bc = ca^6,\ ac = ca^8,\ c^{2018} = b^{2019} \right\rangle.$$

(a) Show that G is finite.

(b) Determine the order of G.

Problem M6. Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix. Assume that $a_{ij} \leq 0$ for $i \neq j$. Show that the following two conditions are equivalent:

- The matrix A is positive-definite.
- There exists a vector v such that both v and Av have strictly positive entries.