Hyperplane Arrangements

Intersection Posets, Characteristic Polynomials, and Regions

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Motivating Questions

Example

If n points are selected from a circle, and all $\binom{n}{2}$ lines joining pairs of the n points are drawn, then what is the maximum number of regions created in the circle?

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The maximum, $\binom{n}{4} + \binom{n}{2} + \binom{n}{0}$, is achieved when no three lines intersect at a point (general position).

Example

What is the maximum number of "regions" determined by n hyperplanes with dimension $d-1$ in $\mathbb{R}^d$?

The maximum, $\sum_{k=0}^{d} \binom{n}{k}$, is achieved when the hyperplanes are taken in general position.

Preliminary Definitions

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For a field K, define an $n-1$ dimensional affine hyperplane of K^n as the affine subspace $\{v \in K^n : a \cdot v = b\}$.

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Definition

The dimension of an arrangement $\mathcal{A}$ in K^n denoted $\dim(\mathcal{A})$ is the integer n. The rank of the arrangement denoted $\text{rank}(\mathcal{A})$ is the dimension of the space spanned by the normals to the hyperplanes.

The Intersection Poset and Characteristic Polynomial

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Define the characteristic polynomial of an arrangement $\mathcal{A}$ as

$$\chi_{\mathcal{A}}(x) = \sum_{s \in L(\mathcal{A})} \mu(V, s) x^{\dim(s)}.$$

Crosscut Theorem

Theorem

For a finite lattice L *and some* $X \subseteq L \backslash \hat{0}$ *such that* $\forall s \in L \backslash \hat{0}$, $\exists t \in X$ *such that* $s \geq t$, *then*

$$\mu(\hat{0}, \hat{1}) = \sum_k (-1)^k N_k,$$

where N_k *is the number of* k*-subsets of* X *whose join is* $\hat{1}$.

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- Then, $\prod_{t \in X} (\hat{0} - t) = \sum_s \mu(\hat{0}, s) s$, and consider the coefficient of $\hat{1}$.

Significance of $\chi_{\mathcal{A}}(x)$

Theorem (Whitney)

For arrangement $\mathcal{A}$ in K^n, then

$$\chi_{\mathcal{A}}(x) = \sum_{B \subseteq \mathcal{A}} (-1)^{\#B} x^{n - rank(B)},$$

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- For any element $t \in L(A)$, $[K^n, t]$ is a lattice.
- Apply the crosscut theorem to $[K^n, t]$ with $X = B$, the set of hyperplanes in $\mathcal{A}$ that contain t.
- Since $\dim(t) = n - \text{rank}(B)$, summing over all t gives the theorem.

Recurrence Relationship for the Characteristic Polynomial

Definition

For a hyperplane $H \in \mathcal{A}$, denote $\mathcal{A} \backslash H$ the arrangement without the hyperplane H. Moreover, denote $\mathcal{A}/H$ the arrangement of nonempty $H \cap J$ in the affine space H for $J \in \mathcal{A}$.

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Let $\mathcal{A}$ be an arrangement in K^n and H be a hyperplane of $\mathcal{A}$. Then, $\chi_{\mathcal{A}}(x) = \chi_{\mathcal{A} \backslash H}(x) - \chi_{\mathcal{A}/H}(x)$.

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- Use Whitney's theorem while considering if H is in B or not.
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- When H is in B, we obtain $(-1) \cdot \chi_{\mathcal{A}/H}(x)$.

Regions and Zaslavsky's Theorem

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For an arrangement $\mathcal{A}$ in $\mathbb{R}^n$, define the number of regions, denoted $r(\mathcal{A})$, to be the number of connected components of $\mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$. Similarly, define $b(\mathcal{A})$ as the number of relatively bounded regions of $\mathcal{A}$.

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$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1),$$
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- $(-1)^n r(A)$ and $(-1)^{\text{rank}(\mathcal{A})} b(A)$ satisfy the same recurrence as $\chi_{\mathcal{A}}(x)$.
- The equations holds when $\mathcal{A} = \emptyset$, and the result follows.

Finite Field Method

A useful method for computing the characteristic polynomial when the hyperplanes are defined over $\mathbb{Q}$.

Theorem

For an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ defined over $\mathbb{Q}$, then for sufficiently large prime power q,

$$\chi_{\mathcal{A}}(q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}} H\right).$$

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- For any $t \in L(\mathcal{A}_q)$ define $f(t) = \#t = q^{\dim(t)}$, $g(t) = \# \left(t - \bigcup_{u>t} u \right)$, and apply the Möbius Inversion Formula

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- Since $f(t) = \sum_{u\geq t} g(u)$, then $g(t) = \sum_{u\geq t} \mu(t,u) q^{\dim(u)}$.
- So, $\chi_{\mathcal{A}}(q) = g(\mathbb{F}_q^n) = \#\left(\mathbb{F}_q^n - \bigcup_{H\in\mathcal{A}} H\right)$.

Interesting Arrangements in $\mathbb{R}^n$

- Braid Arrangement: $x_i - x_j = 0$ and
$$\chi_{\mathcal{A}}(x) = x(x-1)(x-2)\cdots(x-n+1).$$

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