Jacobian groups of biconnected graphs

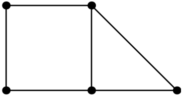
Jeffery Yu

Mentor Dr. Dhruv Ranganathan
Seventh annual PRIMES conference

May 20, 2017

Overview

To every finite graph we associate a group



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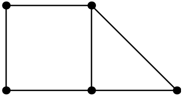

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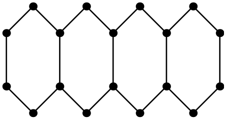
Our goal is to study the effects of forcing a structure to one side on the other

Interactions

- Discrete version of algebraic geometry (tropical Brill-Noether theory)



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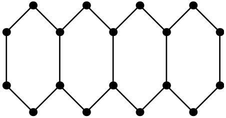
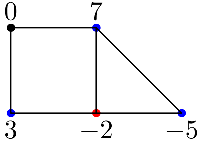


- Random matrices and graphs

Chip configurations

Divisor: Integer linear combinations of vertices



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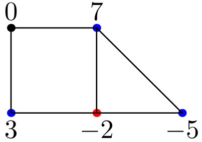
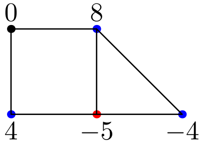


Chip-firing move: Move a chip from the vertex to every adjacent vertex



Jacobian group

Jacobian: Group of all equivalence classes of degree 0

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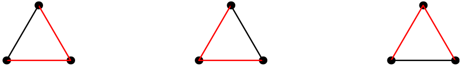

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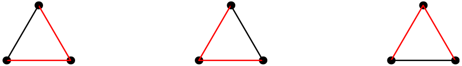

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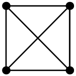
- Finite abelian group: Direct product of cyclic groups



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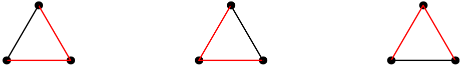

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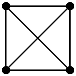

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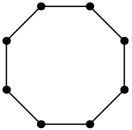
- Torsion group: Every element has finite order, **exponent** is LCM of orders

Jacobian group

Examples:

- The Jacobian of a cycle on n vertices is $\mathbb{Z}/n\mathbb{Z}$



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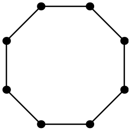

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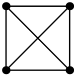

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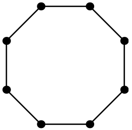

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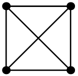
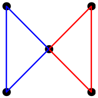


- The Jacobian of a wedge of two graphs is the direct product of the Jacobians of the components.



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Every graph can be decomposed into biconnected components, so focus on biconnected graph

Conjecture

For every positive integer n there exists a nonnegative integer k_n such that for all integers $k > k_n$, the group $(\mathbb{Z}/n\mathbb{Z})^k$ is not the Jacobian of a biconnected graph.

Known results

Gauded-Jensen-Ranganathan-Wawrykow-Weisman (2014): The exponent of a biconnected graph is at least the maximum degree of a vertex in the graph
Corollary: $k_2 = 0$ and $k_3 = 1$

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A large biconnected graph has a vertex of high degree or a long cycle

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High degree vertex and long cycle are opposite ends of spectrum

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Given a graph find a divisor with sufficiently large order

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The exponent of a biconnected graph is at least the length of its longest simple cycle.

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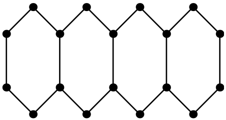
Evidence:

- True for arbitrary genus



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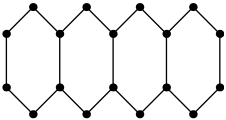
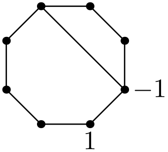


- True if one edge is added to cycle; may lead to inductive approach



Acknowledgments

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- Dr. Tanya Khovanova and the MIT-PRIMES staff
- My parents