Limits of Interlacing Eigenvalues in the Tridiagonal β-Hermite Matrix Model

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Matrices

Recall that an $m \times n$ matrix with entries in $\mathbb{R}$ (or $\mathbb{C}$) is an array of numbers with m rows and n columns.

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Examples

Here are examples of 3×2 and 4×4 matrices:

$$\begin{pmatrix} 3 & -2 \\ e & 1 \\ -\pi & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

Eigenvalues

This is how we multiply a vector by a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix}$$

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Examples

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 34 \end{pmatrix}$$

Eigenvalues

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of a square matrix A if

$$Av = \lambda v$$

for some vector v. It turns out that there are n eigenvalues (up to multiplicity) of an $n \times n$ matrix A.

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Examples

$$\begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

so 3 is an eigenvalue of the original matrix.

Spectral Theorem

Examples

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If a matrix is symmetric, then all of its eigenvalues are real. Generally, we order the eigenvalues as follows:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Random Variables

Define a probability density $p(x)$ to be a function

$$p : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

such that $\int_{\mathbb{R}} p(x)dx = 1$.

Random Variables

A *random variable* X with values in $\mathbb{R}$ and density $p(x)$ is a "random number in $\mathbb{R}$ which can be sampled such that its frequency (histogram) as the number of samples increase converge to $p(x)$."

More precisely,

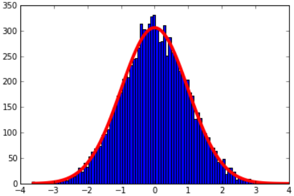
$$\Pr(a \leq X \leq b) = \int_a^b p(x) dx.$$

Example: Gaussian Random Variable

A *Gaussian Random Variable* is one that has

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Here is a sample of 10000 Gaussian random variables with $\mu = 0$ and $\sigma = 1$.



Random Vectors

Define a joint probability density $p(x)$ to be a function

$$p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

such that $\int_{\mathbb{R}^n} p(x) dx^n = 1$.

A random vector is a vector in $\mathbb{R}^n$ that takes random values with joint distribution $p(x)$.

Random Matrices

A random matrix is a matrix whose entries are random variables. Note that the entries do not have to be independent.

We can now consider the eigenvalues of these matrices, etc.

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Examples

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The Model

$$X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0,2) & \chi_{\beta} \\ & & & \chi_{\beta} & \mathcal{N}(0,2) \end{pmatrix}.$$

It turns out that the eigenvalues have joint distribution

$$\frac{1}{Z_n} \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^\beta \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}}.$$

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Motivation. This joint distribution turns out to have an electrostatic interpretation.

Interlacing

We say that two sequences $\{x_i\}_{i=1}^{n}$, $\{y_j\}_{j=1}^{n-1}$ interlace if

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Let A be a symmetric n by n square matrix, and let A' be its $n-1$ by $n-1$ lower right submatrix.

Examples

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad A' = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

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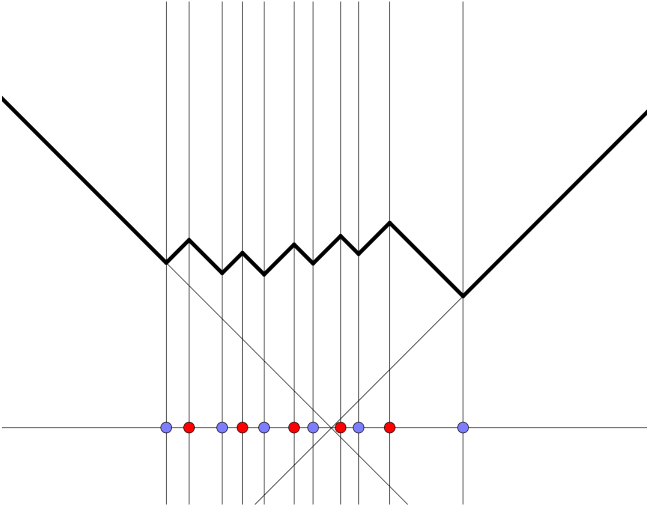
Examples

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad A' = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

The eigenvalues of A and A' interlace.

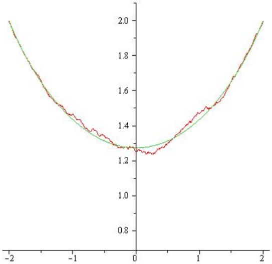
What We Are Doing With the Model

Let X_{n-1} be the lower $n-1$ by $n-1$ submatrix of X_n. We saw that the eigenvalues of X_n and X_{n-1} interlace:



What We are Doing With the Model (continued)

As $n \to \infty$, these diagrams converge to some curve:



We are interested in what this “limiting shape” is.

Method of Traces

It turns out that the study of these diagrams is equivalent to considering what happens to

$$\mathrm{tr} X_n^k - \mathrm{tr} X_{n-1}^k$$

as $n \to \infty$. We can work with the trace combinatorially:

$$\mathrm{tr} X_n^k - \mathrm{tr} X_{n-1}^k = \sum_{\vec{i} \in \mathcal{B}_k} \prod_{j=1}^{k} X_n(i_j, i_{j+1})$$

where

$$\mathcal{B}_k = \{(i_1, \ldots, i_k) \in [n]^k : |i_j - i_{j+1}| \leq 1 \text{ and } \exists i_j = 1\}.$$

Current Results

Main Theorem

In the β-Hermite case, the diagrams converge to the Logan-Shepp curve:

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x \arcsin(\frac{x}{2}) + \sqrt{4-x^2}), & |x| \leq 2 \\ |x|, & |x| \geq 2 \end{cases}$$

We have also shown that the fluctuations of the diagrams from the curve are gaussian in some sense.

Future Work

We also want to look at other random matrix models, such as β-Laguerre and β-Jacobi. For β-Laguerre, we can describe the limiting shape, but we conjecture that the fluctuations of the diagrams from the curve are not gaussian.

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