A Generalization of Erdös-Szekeres to Permutation Pattern Replacement

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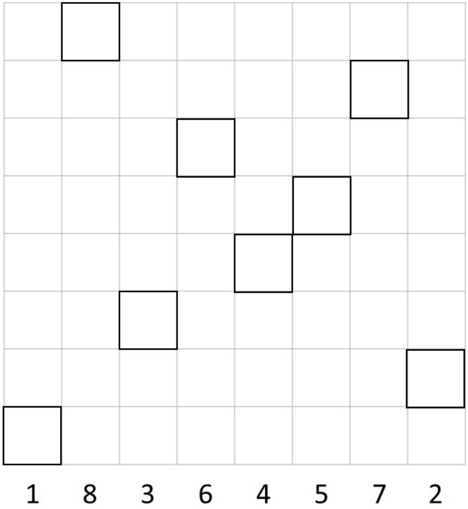
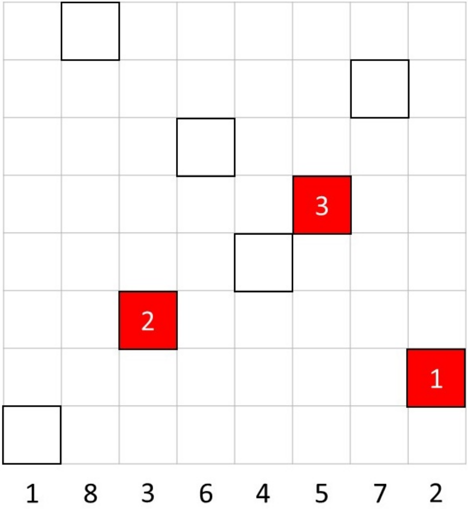
What is a Permutation?



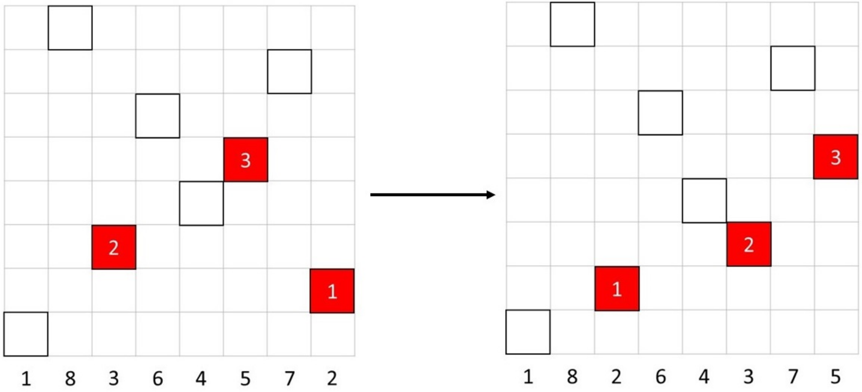
What is a Pattern?

Example: 231-pattern



Pattern Replacement

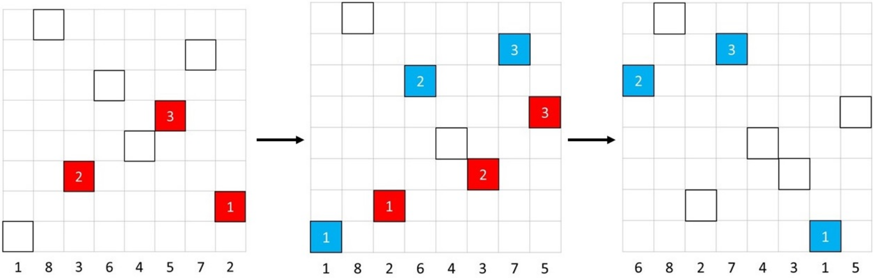
Rearranging the numbers in one pattern to form another.
Example: 231 → 123



Pattern Replacement Equivalence Relations

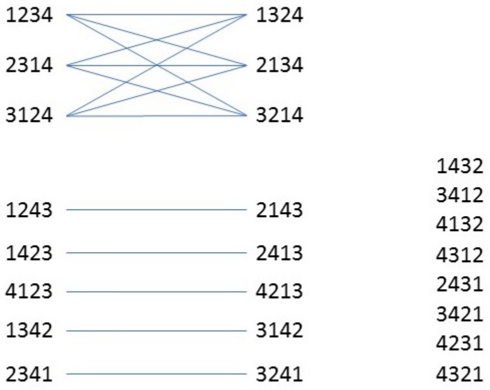
Permutations are equivalent if they can be reached through a series of pattern replacements.

Example: $(231 \leftrightarrow 123)$-equivalence relation



Equivalence Classes for $(213 \leftrightarrow 123)$ relation

Pattern replacements connect permutations together in a graph:



Each connected component is an *equivalence class*.

Directions of Research

Two Main directions of Past Work
Studying Individual Equivalence Relations

- **Example:** The number of equivalence classes for $(213 \leftrightarrow 123)$-equivalence is C_n, the n^{th} Catalan number.

Studying Infinite Families

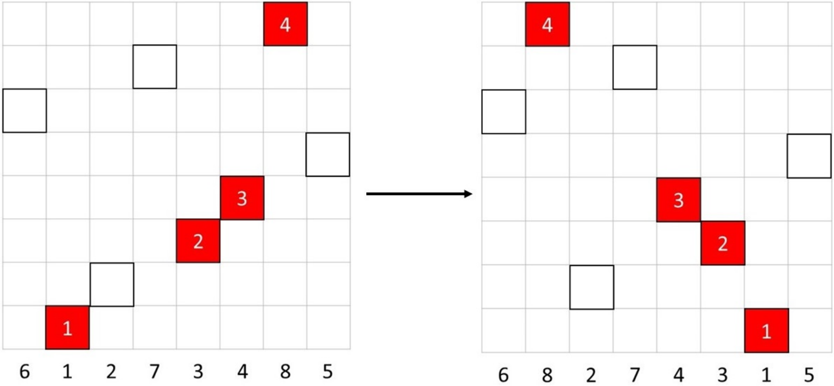
- **Example:** Rotational pattern replacements

  E.g. $(1234 \leftrightarrow 2341 \leftrightarrow 3412 \leftrightarrow 4123)$-equivalence

Our Research: We study a new infinite family:

The $(123 \cdots (k-1)k \leftrightarrow k(k-1) \cdots 321)$-equivalence.

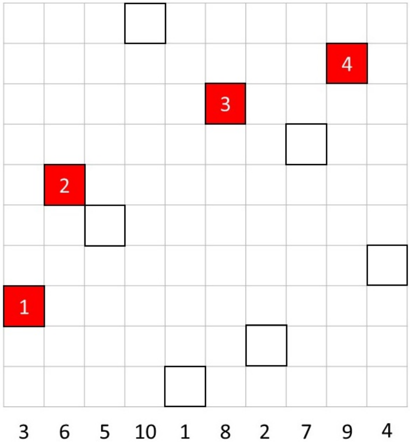
EXAMPLE: $1234 \leftrightarrow 4321$ REPLACEMENT.



What makes this interesting?

Erdös-Szekeres Theorem: If $n \geq k^2 - 2k + 2$, every permutation of length n must contain a $123 \cdots (k-1)k$ or a $k(k-1) \cdots 321$ pattern.

Example: Perms of length 10 must contains 1234 or 4321.



A GENERALIZATION OF ERDÖS-SZEKERES

Our Theorem:

- Consider $(123\cdots(k-1)k \leftrightarrow k(k-1)\cdots 321)$-equivalence relation on permutations of length n.
- If $n \geq 3k^2 - 4k + 3$, then

$$\#\text{ equivalence classes } = \begin{cases} 2, \text{ if } k \equiv 0 \mod 4 \\ 2, \text{ if } k \equiv 1 \mod 4 \\ 1, \text{ if } k \equiv 2 \mod 4 \\ 1, \text{ if } k \equiv 3 \mod 4 \end{cases}$$

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- In the first two cases, the even and odd permutations are in separate classes.

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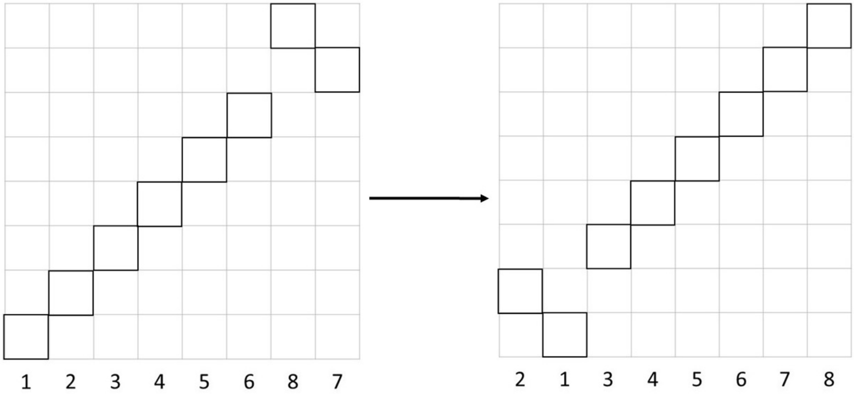
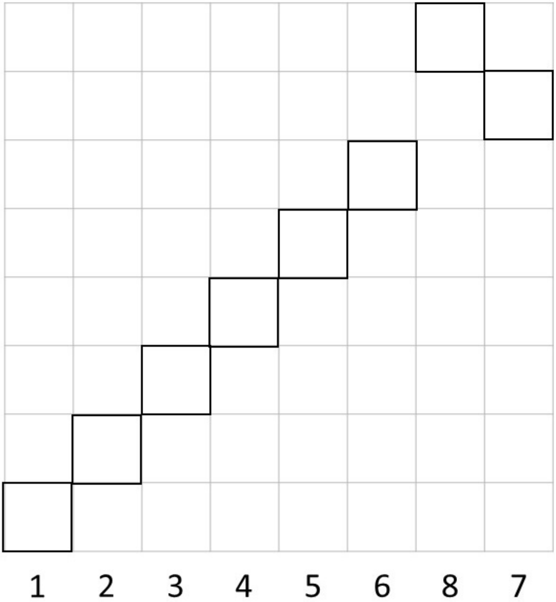
A Key Observation

For $n \geq 2k$, the $(123 \cdots k \leftrightarrow k \cdots 321)$-equivalence makes the permutations $123 \cdots (n-2)n(n-1)$ and $213 \cdots (n-2)(n-1)n$ equivalent.

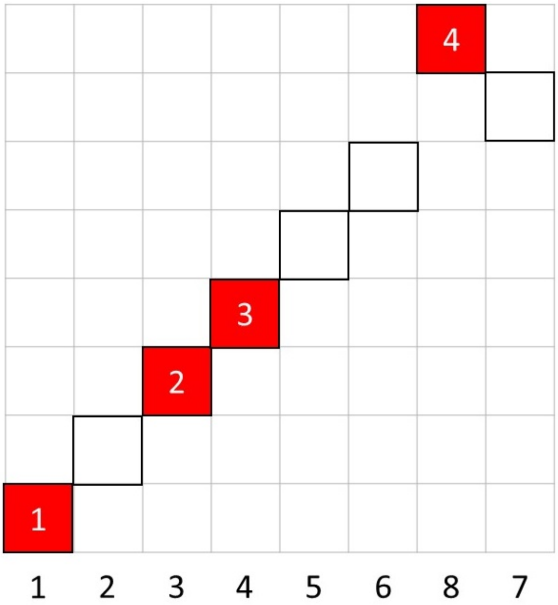
Example: $k = 4$**:**



PROOF



Proof



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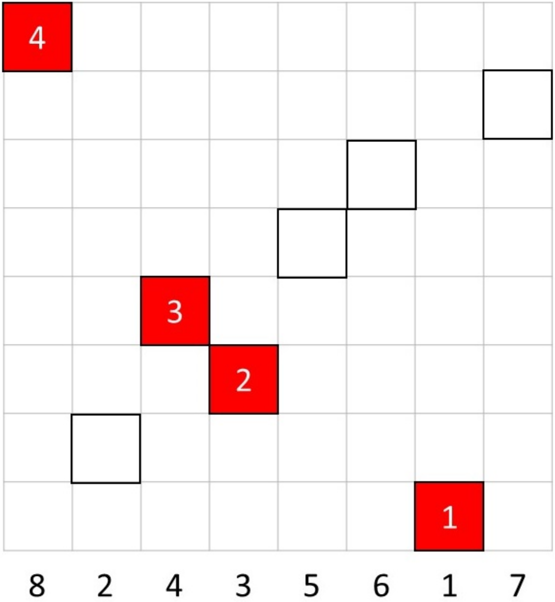

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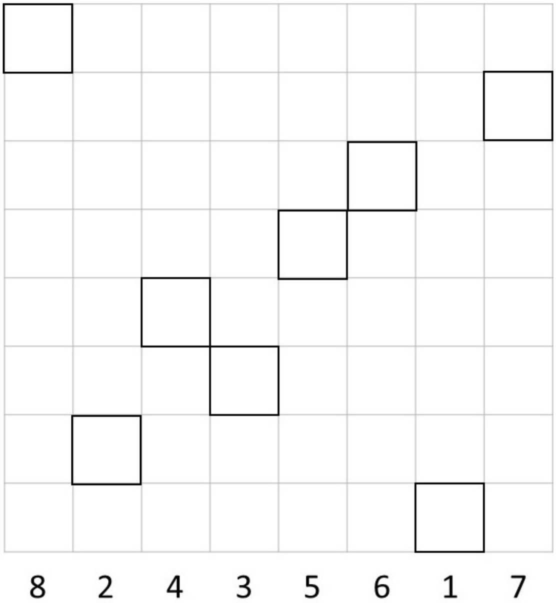

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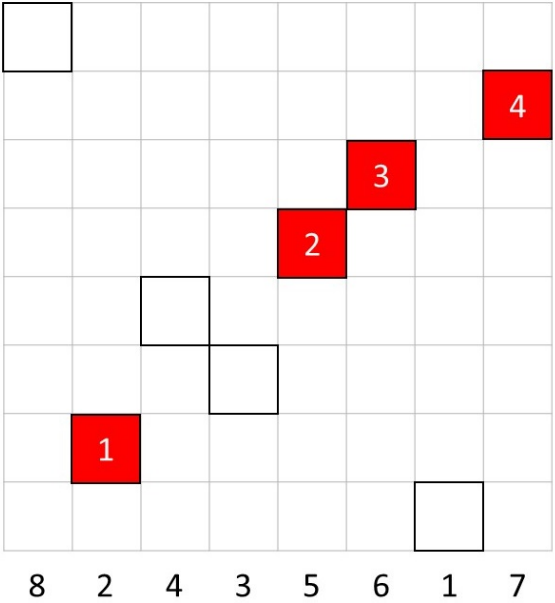

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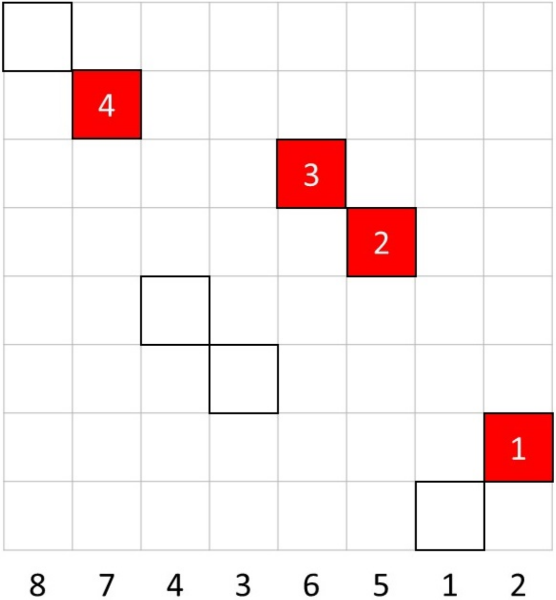

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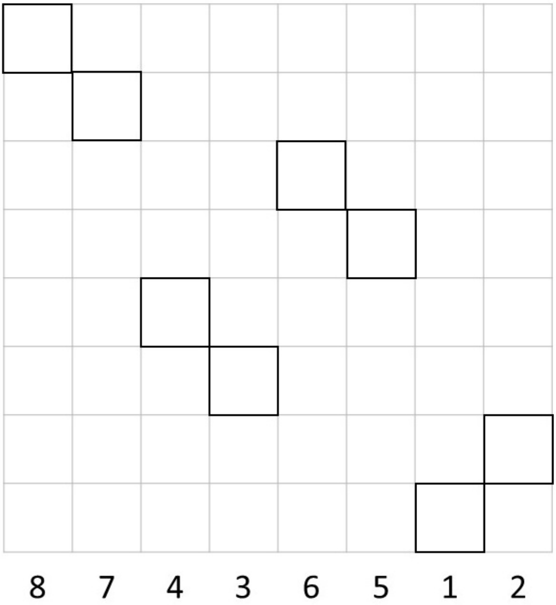

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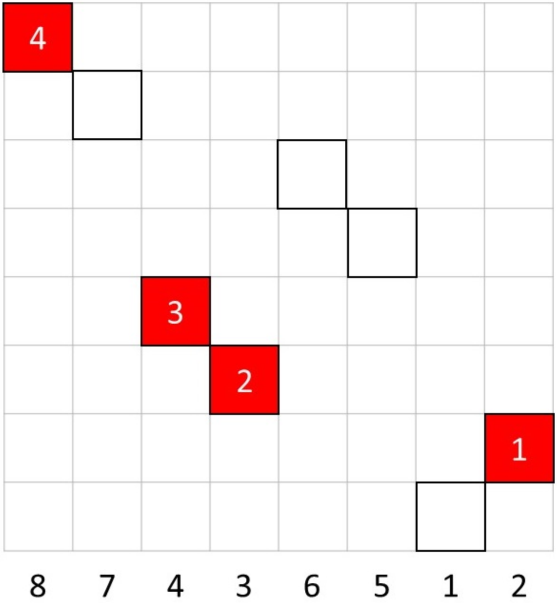

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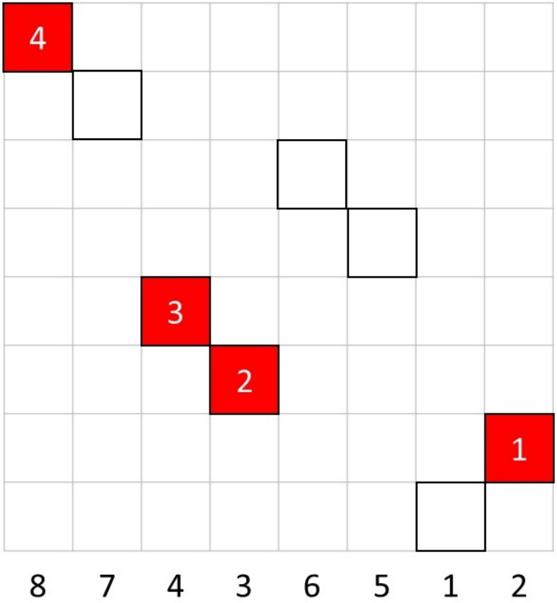

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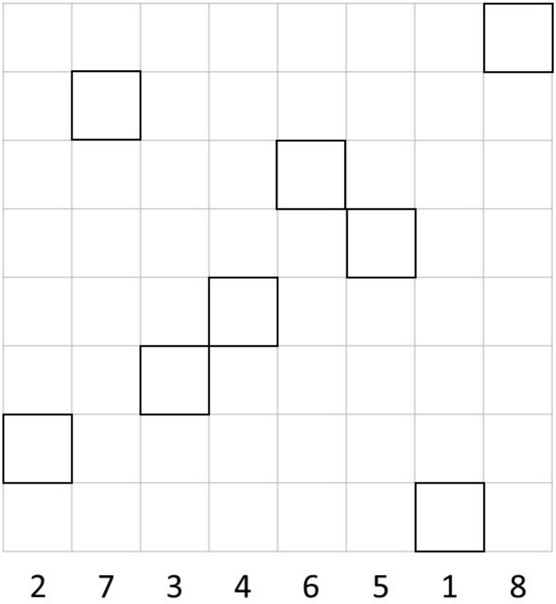

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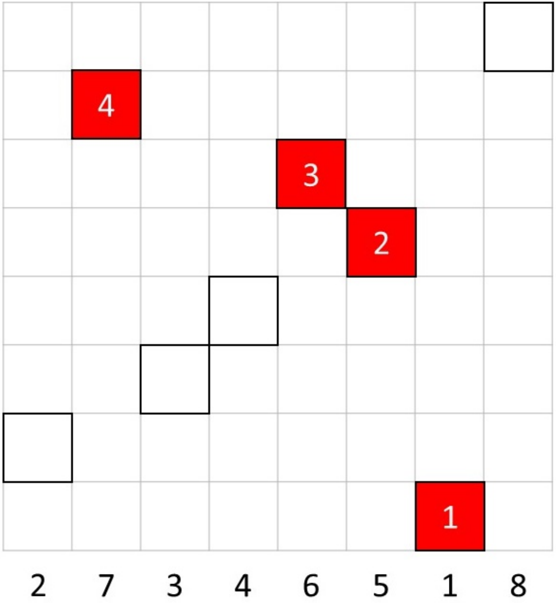

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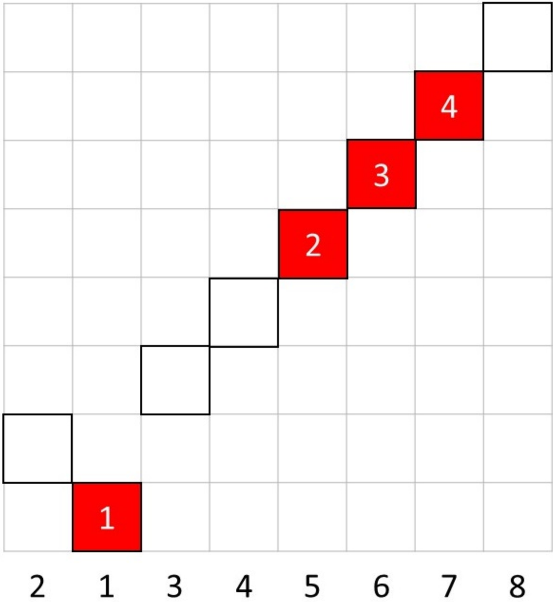

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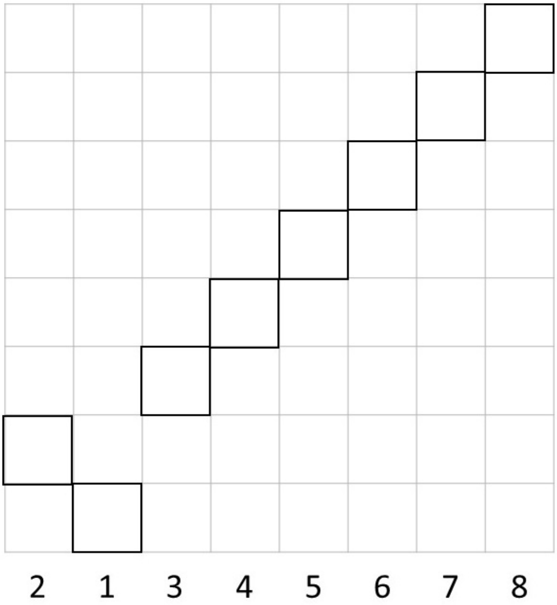

Future Work

- **Improving our bounds:** Our theorem holds for $n \geq 3k^2 - 4k + 3$. Erdös-Szekeres Theorem holds for $n \geq k^2 - 2k + 2$. Can we close the gap?
- **Individual Patterns:** Past researchers considered relations with patterns of length 3.
  What can we say for patterns of length 4?
- **Other Infinite Families:** Kuszmaul and Zhou pose several open problems on cyclic pattern-replacement relations.

Acknowledgments

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- Head PRIMES mentor Tanya Khovanova for helping me prepare this talk.
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References

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