Pattern Avoidance Classes Invariant Under the Modified Foata-Strehl Action

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Permutations

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Example

The string $\pi = 67284135$ is a permutation of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Peaks and Valleys

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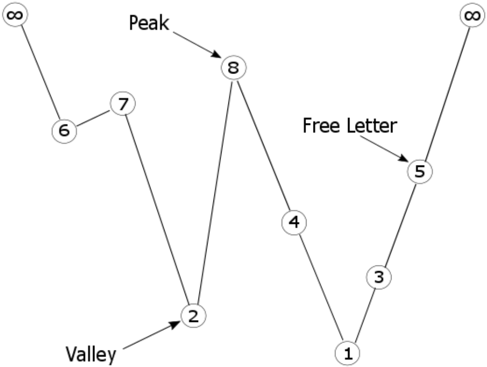
The "mountain range" representation of the permutation 67284135:



Pattern Avoidance

- Consider a "long" permutation, such as $\pi = 23514$ and a shorter one, say $\sigma = 132$.
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- We say the permutation π *contains* σ.
- The permutation 15234 *avoids* the pattern 321.

Pattern Avoidance Classes

- Let π be a permutation of $[n]$ and let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be a set of patterns each of length at most n. We say that π avoids Σ if π avoids every pattern in Σ.

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- Let π be a permutation of $[n]$ and let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be a set of patterns each of length at most n. We say that π avoids Σ if π avoids every pattern in Σ.
- $Av_n(\Sigma)$ denotes the set of all length-n permutations p such that p avoids Σ.
- $Av(\Sigma)$ denotes the set of all permutations that avoid Σ.

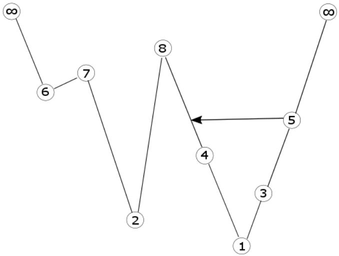
Valley Hopping

Definition

A *valley-hop* (formally known as the modified Foata-Strehl Action) $H_j(\sigma)$ is the permutation obtained by moving the free letter j in σ across the adjacent valleys to the nearest slope of the same height.

Example

$H_j(\sigma)$ for $j = 5$ and $\sigma = 67284135$



Hop Equivalence Classes

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Two permutations σ_1 and σ_2 of $[n]$ are in the same *hop equivalence class* if there exists some sequence of valley-hops $H_{i_1}, H_{i_2}, \ldots, H_{i_k}$ such that $H_{i_1}(H_{i_2}(...(H_{i_k}(\sigma_1))...)) = \sigma_2$. We let $Hop(\sigma)$ denote the hop equivalence class of σ.

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Example

$Hop(13542) = \{13542, 13524, 31542, 31524\}$

Valley Hopping and Pattern Avoidance Classes

Definition

Let Σ be a set of patterns. We say that $Av_n(\Sigma)$ is invariant under valley-hopping if for any permutation $\pi \in Av_n(\Sigma)$, any valley-hop π' of π is also in $Av_n(\Sigma)$.

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- If $Av_n(\Sigma)$ is invariant under valley-hopping for all n, the distribution of peaks and valleys for permutations in $Av(\Sigma)$ is well understood.
- Our problem: Classify all pattern sets Σ such that $Av(\Sigma)$ is invariant under valley-hopping.

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Proposition

Let Σ be a nontrivial singleton pattern set with $Av(\Sigma)$ invariant under valley-hopping. Then $\Sigma = \{132\}$ or $\Sigma = \{231\}$.

Single Hop Equivalence Classes

- If Σ is a pattern set such that $Av(\Sigma)$ is invariant under valley-hopping and $\sigma \in \Sigma$, then $Hop(\sigma) \subseteq \Sigma$.

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- Only nontrivial Σ with more than one element for which $Av(\Sigma)$ was known to be invariant under valley-hopping was $\Sigma = \{1423, 1432\}$.

Single Hop Equivalence Classes

- If Σ is a pattern set such that $Av(\Sigma)$ is invariant under valley-hopping and $\sigma \in \Sigma$, then $Hop(\sigma) \subseteq \Sigma$.
- Only nontrivial Σ with more than one element for which $Av(\Sigma)$ was known to be invariant under valley-hopping was $\Sigma = \{1423, 1432\}$.
- Can we classify all σ for which $Av(Hop(\sigma))$ is invariant under valley-hopping?

Classification for Single Hop Equivalence Classes

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Proposition

There does not exist a position $i < |\sigma|$ for which i and $i+1$ are both free letters in σ.

Classification for Single Hop Equivalence Classes (Cont.)

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Av of the hop equivalence classes for following permutations are invariant under valley-hopping:

- 132, 231
- 1423, 2413, 3412, 1243, 1342, 2341
- 12534, 13524, 14523, 23514, 24513, 34512

Construction for General Pattern Sets

- For any permutation pattern σ of length n, there is a trivial pattern set Σ such that $Av_n(\Sigma)$ is invariant under valley-hopping. We would like to improve upon this:

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Theorem

Let σ be a permutation pattern of length n. Then there exists a family of length-n permutation patterns Σ containing σ such that

$$|\Sigma| < \frac{n!}{(n - pk(\sigma))!} 2^{n-2pk(\sigma)-1}$$

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- Improvement over trivial family size of $n!$ by factor of $\frac{(n-pk(\sigma))!}{2^{n-2pk(\sigma)-1}}$.

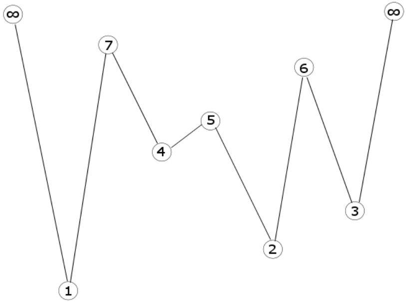
Alternating Permutations

Definition

A permutation is *strictly alternating* if it has no free letters.

Example

1745263 is a strictly alternating permutation.



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Theorem

Let Σ be a family of strictly alternating permutations with k peaks such that $Av(\Sigma)$ invariant under valley-hopping. Then there exists some subset Π of Σ of size $(k-1)!$ such that

- Every permutation in Π is tall
- Every permutation in Π has the same valleys
- For any $\pi \in \Pi$, the letter in position 2 is the smallest peak.

Future Work

- Classify all sets of strictly alternating permutations with $Av(\Sigma)$ invariant under valley-hopping.
- Current strategy: start with singleton pattern set Σ and insert more elements until $Av(\Sigma)$ is invariant under valley-hopping. Is there a more general way of generating Σ invariant under valley-hopping?

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