Multi-Crossing Numbers for Knots

Grace Tian

The Wellington School

Mentor: Jesse Freeman

Abstract

We study the projections of a knot K that have only n-crossings. The n-crossing number of K is the minimum number of n-crossings among all possible projections of K with only n-crossings. We obtain new results on the relation between n-crossing number and $(2n-1)$-crossing number for every positive even integer n.

1 Introduction

Knots have been around for thousands of years, but they have only attracted the attention of mathematicians in the last 150 years [7]. In 1867, the physicist William Thomson [10] hypothesized that atoms were knots in a medium called the ether. As a result, many scientists were motivated to study knots and mathematicians began to classify knots in tables. Although the Michelson-Morley experiment of 1887 [9] dismissed Thomson's atom/knot hypothesis, knot theory has over time become a promising field of mathematical research in its own right.

A knot is a closed curve in $\mathbb{R}^3$ that is homeomorphic to a circle [7]. Two knots are equivalent if one can continuously deform one knot into the other knot.

An effective way to study knots is to consider the projection of a knot onto a plane. The knot diagram of a knot is a projection of the knot with additional information about overcrossing or undercrossing at each crossing point in the diagram. In 1927, Reidemeister [8] showed that two knot diagrams represent the same knot if and only if they are related by a finite sequence of Reidemeister moves, which consist of a twist move, a poke move, and a slide move.

Traditionally, mathematicians have only studied regular knot projections, in which every crossing of a knot has two strands. The study of multi-crossing projections of knots was initiated by Adams in papers [1, 2, 3] around 2010 to go beyond regular projections.

A n-crossing is a crossing in a projection of a knot or link that has n strands that bisect the knot. A n-crossing projection is a projection such that all of the crossings are n-crossings. For each crossing, we identify the heights of the strands in an n-crossing with integers $1, 2, \ldots, n$, where $i > j$ indicates that strand i crosses over strand j. In 2013, Adams proved that given any integer $n \geq 2$ and any knot, there exists a n-crossing projection [1]. The n crossing number of a knot or link K, denoted by $c_n(K)$, is defined to be the minimum number of n-crossings among all possible n-crossing projections of K. Adams further showed that $c_n \geq c_{n+2}$ for all $n \geq 2$ by local moves at each crossing [6]. In other words, $c_n(K)$ of the same parity forms a decreasing sequence. He also showed that $\frac{c_2(K)}{3} \leq c_3(K) \leq c_2(K) - 1$.

The motivation to study the multi-crossing number $c_n(K)$ lies in the fact that it is a topological invariant that can distinguish $5_1 \# 3_1$ from $\overline{5_1} \# \overline{3_1}$ [4]. There is a chance that $c_n(K)$ can distinguish mutant knots, which no polynomial invariant can distinguish.

In this paper, we show that for any positive even integer n and any knot K, $c_n(K) \geq c_{2n-1}(K)$. We are able to turn an even n-crossing projection into a

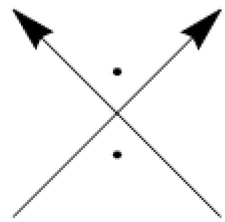

Figure 1: An orientation induced state of a double crossing. [6]

$(2n-1)$-crossing projection by a series of global and local moves. This is the first general result on the relation between $c_n(K)$ with different parities.

2 $C_n(K) \geq C_{2n-1}(K)$

A *state marker* at a given crossing is a pair of dots placed at the corners of two opposite regions near the crossing [6]. If the projection D is oriented, then we may use the orientation to position a state marker at each crossing as illustrated in Figure 1. We place a state marker in between two strands if the two adjacent strands agree in orientation. If we label the state marker at every crossing by using the orientation, we call this the *orientation induced state* of D.

Lemma 1. *For any even crossing, there is an odd number of state markers. For any odd crossing, there is an even number of state markers.*

Proof. Let there be an orientation induced n-crossing. We bisect the given n crossing and select one of the halves. Let the kth strand be k strands away from the bisecting line in the counterclockwise direction. If the kth strand agrees with the previous strand in the counterclockwise direction, then let $x_k = 0$. If the same pair does not agree, then let $x_k = 1$.

Thus, we have

$$x_1 + x_2 + \cdots + x_n \equiv 1 \pmod 2.$$

If n is even, then there must be an odd number of zeros, and thus an odd number of state marker pairs. If n is odd, there must be an even number of zeros, and thus an even number of state marker pairs. □

This immediately gives

Corollary. *Every orientation induced even crossing has state markers.*

A version of the lemma below has been shown in [6] for regular projections. We slightly modify the proof to generalize for any n-crossing projection.

Lemma 2. *Any orientation induced state of multi-crossing knot projection has an even number of state markers in each region.*

Proof. Let R be a complementary region of K. If R is a multi-connected domain, then split the vertex or vertices so that each strand connects two different vertices and introduces no extra crossings. As we traverse the boundary of R, we encounter an even number of vertices where the orientation of the edges on ∂R reverses from clockwise to counter-clockwise or vice versa. These are exactly the vertices that contribute a dot to R. Thus R contains an even number of dots and each region has an even number of state markers. □

Given a knot diagram D, a *crossing circle* for D is a circle C embedded in the projection plane such that C meets D transversely in some set of crossings of D [6]. We use the orientation induced state to form crossing circles of D.

Lemma 3. *Every even crossing projection of a knot or link has at least one crossing circle.*

Proof. Because of Lemma 2, we can connect two state markers within any region that has state markers. Then, within that region, we can connect the state marker to its opposite pair through the crossing. Connecting all the state markers in this manner forms at least one crossing circle. □

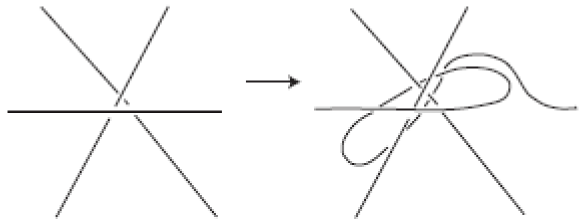

Figure 2: Converting a 3 crossing to a 5 crossing [6]

In [6], Adams made the observation that any n-crossing can always be increased to an $(n+2)$-crossing. After passing through the crossing on any of the n strands, we can locally loop back and forth underneath the crossing two more times before continuing on as before. This process is shown for turning a 3-crossing into a 5-crossing in Figure 2. We use this process on the remaining crossings that do not have $(2n-1)$ strands when we turn an even n-crossing projection into a $(2n-1)$-crossing projection in the following theorem.

Theorem 4. *For any positive even integer n and any knot K, $c_n(K) \geq c_{2n-1}(K)$.*

Proof. Assume that K is a minimal n-crossing projection, where n is an even integer. We choose a strand that is adjacent to its crossing circle by Lemma 3. We can pull that over stand along the crossing circle, as shown in the third image of Figure 3. We do this for each crossing circle. Because of Lemma 1, there is now an odd number of strands of at most $2n-1$ at each crossing. As desired, we add an extra even number of crossings so that each crossing has $2n-1$ strands. We do so by locally looping back and forth so that an n crossing becomes a $n+2$ crossing as shown in Figure 2. We now have a $(2n-1)$-crossing projection with the same number of crossings as our even n crossing projection. Thus, $c_n \geq c_{2n-1}$. $\square$

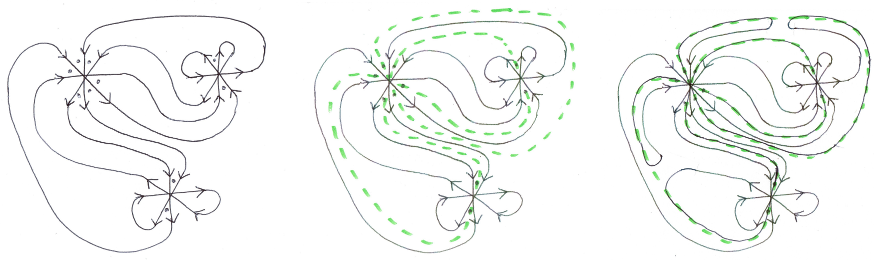

Figure 3: The steps to create a 4-crossing projection into a 7-crossing projection via crossing covering circles. To form a 7-crossing projection, local moves in Figure 2 are needed to add strands to the top right and bottom crossings.

Remark. Most even n-crossing projections can be turned into a odd crossing projection with fewer than $2n-1$ crossings. If there is only one crossing circle in a even n-crossing projection of a knot K, then $c_n(K) \geq c_{n+1}(K)$.

Corollary. *For any knot K, $c_n(K) \geq c_{2n-1}(K)$.*

Proof. Because $c_n \geq c_{n+2}$ [6], the corollary holds for odd n. □

3 Conclusions and future directions

We developed a method to turn any even crossing projection to an odd crossing projection. We used this method to show that $c_n(K) \geq c_{2n-1}(K)$ for any positive even integer and any knot K.

One possible direction for future research would be to develop a general method to convert an odd crossing projection to an even crossing projection, hopefully losing one crossing in the process. With this, we would be able to show that $c_n(K) > c_{2n-1}(K)$ or $c_n(K) > c_{2n}(K)$ for odd integer n. Even crossing projections pose new difficulties, however, since not every odd crossing projection has state markers and crossing covering circles necessary to add extra crossings. It is important to consider local crossing cases rather than global cases (i.e. with crossing circles) for odd crossing projections.

Another possible direction for future research is to show that the crossing number is monotonic, i.e. $c_n(K) \geq c_{n+1}(K)$ for every positive integer n. The computer generated results of [4] suggest that the multi-crossing numbers up to c_9 of all the prime knots with at most 9 crossings have been monotonic.

Finally, it would be useful to improve the results of [5] and find additional bounds on übercrossing and petal numbers for knots.

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