The PRIMES 2016 Math Problem Set

Dear PRIMES applicant!

This is the PRIMES 2016 Math Problem Set. For complete rules, see
`http://math.mit.edu/research/highschool/primes/apply.php`
(for MIT PRIMES), and
`http://math.mit.edu/research/highschool/primes/usa/apply-usa.php`
(for PRIMES-USA).

Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

WARNING: Posting these problems on problem-solving websites is strictly forbidden! Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General math problems

Problem G1. Let N be a positive integer. A soon to be bankrupt casino lets you play the game $G(N)$. In the game $G(N)$, you roll a typical, fair, six-sided die, with faces labeled 1 through 6, up to N times consecutively. After each roll, you may either end the game and be paid the square of the most recent number you rolled, or roll the die again hoping for a better number — on the N-th roll you must take the money and cannot roll again. For example, in the game $G(2)$ you might first roll a 5, but, hoping for a 6, you roll again, only to be disappointed to roll a 1 on your second and final roll, and you walk away with $1.

(a) Describe a strategy that maximizes the expected value of playing $G(N)$.

(b) What is this maximal expected value?

Problem G2. (a) Let n be an even positive integer. Can one divide the numbers $1, ..., n$ into three nonempty groups, so that the sum of numbers in the first group is divisible by $n+1$, in the second one by $n+2$, and in the third one by $n+3$?

(b) For which odd positive integer numbers n can one do this?

Problem G3. Suppose you play a game whose goal is to collect three cards of the same suit. In your first move, you take three cards from a standard 52-card deck at random. Call them $C1, C2, C3$.

1. If $C1, C2, C3$ are all of the same suit, you win.

2. If $C1, C2, C3$ are all of different suits, you put them back, shuffle, and take three cards one more time. If now all are of the same suit, you win, otherwise, you lose.

3. If among $C1, C2, C3$, exactly two cards are of the same suit, you put the third card (the odd one out) back into the deck, shuffle, and pull out a card. If it is the same suit as the other two, you win, otherwise, you lose.

What is the chance of winning? (Write the answer as a fraction in lowest terms).

Problem G4. In a couples therapy session, n couples are to be seated at a round table (in $2n$ chairs), but no person is allowed to sit next to his/her spouse. How many seat assignments are there? What is the number of seatings for 5 couples?

Problem G5. A zero-one matrix A is said to *contain* another zero-one matrix P if P is a submatrix[1] of A, or some submatrix of A can

[1] A submatrix is obtained from a matrix by crossing out some rows and some columns.

be transformed to P by changing some ones to zeroes. Otherwise A is said to *avoid* P.

Consider the following pattern avoidance game, denoted by PAG(n, P): Starting with the $n \times n$ all zeroes matrix, two players take turns changing zeroes to ones. If any player's turn causes the matrix to contain the pattern P, then that player loses.

If no dimension of P exceeds n, then PAG(n, P) will always have a winner. Define $W(n, P)$ to be the winner of PAG(n, P) if both players employ optimal strategies.

(a) Determine $W(n, P)$ for every $n \geq k$ when P is a k by 1 matrix with every entry equal to 1.

(b) Determine $W(n, P)$ for every $n \geq 2$ when P is a 2 by 2 identity matrix: $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Problem G6. Suppose that n pine trees grow at points $T_1, ..., T_n$ of the plane (no three on the same line). A cyclic order C of $T_1, ..., T_n$ (i.e., an order up to cyclic permutation) is called *visible* if there exists a point P in the plane from which an observer sees the trees $T_1, ..., T_n$ in the order C. The observer has a 360 degree vision, starting at an arbitrary angle and sweeping clockwise. Observation points are such that no two trees are on the same line of vision. The positions and labeling of the trees are fixed. E.g. if there are 4 trees, and tree 1 in the East, tree 2 in the West, tree 3 in the North, and tree 4 in the South from the observer then the order is 1423, or any cyclic permutation of these (e.g. 3142).

Show that if $n \geq 7$ then there exists a cyclic order which is not visible. What about $n = 6$?

Problem G7. A permutation s of n elements has order 2016 (i.e., the smallest number of times you need to repeat s to get to the original position is 2016). What is the smallest possible value of n? Give an example of such s for the minimal n. (*Hint: consider the cycle decomposition of* s).

Advanced math problems

Problem M1. There are n piles with coins. In one move you can pick several piles and take the same number of coins from those piles. Given a set of piles, its *piles number* is the smallest number of moves you need to remove all coins from all the piles. For example, if you have three piles with 1, 2, and 3 coins each, you can remove all the coins in three moves by treating one pile at a time. But the piles number is 2, as the smallest number of moves is 2.

Find the piles number (with proof) for the following sets of piles:

- 1, 2, 3, 10, 20, 30, 100, 200, 300.
- 1, 2, 3, 11, 12, 13, 101, 102, 103.
- 1, 3, 4, 7, 11, 18, 29, 47, 76, 123.
- Any sequence of natural numbers of length n where each term starting from the third one is the sum of two previous terms.

Problem M2. Let $f_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$.

(a) Show that $f_n(x) > 0$ for all real x if n is even, and that f_n has a unique real root x_n for n odd.

Hint: use the relationship between f_n and its derivative.

(b) Show that all complex roots of f_n are simple (i.e., if a is a root of f_n then $f_n'(a) \neq 0$).

(c) Let $n = 2k + 1$ for positive integer k, and $c = \lim_{k\to\infty} \frac{x_n}{n}$ (it is known to exist). Find c. (Represent the answer as a root of an equation, and compute it to 4th digit precision). *Hint: Use the relationship between $f_n(x)$ and the function e^x.*

Problem M3. Let p be a prime.

(a) Find the number of square matrices A of size n over the field $\mathbb{F}_p$ of p elements such that $A^p = A$.

(b) Suppose that $p \geq 3$. Find the number of square matrices of size n over $\mathbb{F}_p$ such that $A^2 + 1 = 0$ (where 1 is the identity matrix and 0 is the matrix of all zeros). You may have to consider two cases for p.

Problem M4. Suppose we are given integers $m, n > 0$, and a collection S of (distinct) subsets of some ambient set $\mathcal{A}$, each of size at most m. Assume $|S| > (n-1)^m m!$. Prove that there exist n sets $A_1, \dots, A_n \in S$ such that the intersections $A_i \cap A_j$ are the same for all pairs (i, j) with $i \neq j$.

Problem M5. Find the number of colorings of the faces of the cuboctahedron (`https://en.wikipedia.org/wiki/Cuboctahedron`) in n colors, up to rotations (i.e. two colorings equivalent by rotation are regarded as the same).

Problem M6. Let $D_i, i \geq 1$ be open disks of radii $r_i < 1$ contained in the unit disk D, such that $D = \cup_{i\geq 1} D_i$.

(a) Show that for each $0 < a < 1$ the series $\sum_i r_i^a$ is divergent.

(b) Show that $\sum_i r_i$ is divergent.

(c) For any $a > 1$, can you pick D_i so that $\sum_i r_i^a$ is convergent?

(d) Can you solve (a),(b) if the union of the disks D_i is not necessarily the whole D but a subset $D' \subset D$ of full area (i.e., area π)?

Hint. Consider the intersection of D_i with the circle of radius $1 - t$ centered at the origin, or (for (d)) the annulus between this circle and the unit circle.