DIAGRAMMATIC CALCULUS OF COXETER AND BRAID GROUPS

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Abstract. We investigate a novel diagrammatic approach to examining strict actions of a Coxeter group or a braid group on a category. This diagrammatic language, which was developed in a series of papers by Elias, Khovanov and Williamson, provides new tools and methods to attack many problems of current interest in representation theory. In our research we considered a particular problem which arises in this context. To a Coxeter group W one can associate a real hyperplane arrangement, and can consider the complement of these hyperplanes in the complexification Y_W. The celebrated $K(\pi, 1)$ conjecture states that Y_W should be a classifying space for the pure braid group, and thus a natural quotient ${}^{Y_W}/_W$ should be a classifying space for the braid group. Salvetti provided a cell complex realization of the quotient, which we refer to as the Salvetti complex. In this paper we investigate a part of the $K(\pi, 1)$ conjecture, which we call the $K(\pi, 1)$ conjecturette, that states that the second homotopy group of the Salvetti complex is trivial. In this paper we present a diagrammatic proof of the $K(\pi, 1)$ conjecturette for a family of braid groups as well as an analogous result for several families of Coxeter groups.

1. Introduction

Group theory, which is the study of algebraic structures known as groups, is a vitally important part of mathematics that has applications in various fields including physics, chemistry, crystallography, cryptography, and combinatorics, as well as being a rich area of study in its own right [2, 8, 9, 11, 1, 7]. Two groups that arise often in the study of natural phenomenon are the dihedral group D_n and the symmetric group S_n. The dihedral group and symmetric group are both special cases of a more general class of groups known as Coxeter groups—the main focus of our project.

In addition to being generalizations of the natural reflection groups, Coxeter groups have a myriad of uses in mathematics, especially in representation theory, where they serve as building blocks for the classification of algebraic objects. Examples of finite Coxeter groups include the symmetry groups of regular polytopes and the Weyl groups of simple Lie algebras, which are very important in the study of particle physics [11]. Infinite Coxeter groups include symmetry groups of regular tessellations of Euclidean space and Weyl groups of affine Kac-Moody algebras, which are a generalization of semisimple Lie algebras and are of particular importance in conformal field theory and the theory of exactly solvable models [7, 11].

To a Coxeter group W one can associate a real hyperplane arrangement and consider the complement of these hyperplanes in the complexification Y_W. The celebrated $K(\pi, 1)$ conjecture in modern algebraic topology states that Y_W should be a classifying space for the pure braid group, and thus a natural quotient ${}^{Y_W}/_W$ should be a classifying space for the braid group. In [10], Salvetti provided a cell complex realization of the quotient, which we refer to as the *Salvetti complex*. The $K(\pi, 1)$ conjecture was proven for finite Coxeter groups by Deligne in [3] but many cases remain open. In this paper we use a novel approach to investigate a part of the $K(\pi, 1)$ conjecture, which we refer to as the $K(\pi, 1)$ conjecturette, that states that the second homotopy group (denoted as π_2) of the

Salvetti complex is trivial. In [4], another cell complex was introduced as a 3-skeletal model for the classifying space of a Coxeter group W. In this paper, we also prove that π_2 of this cell complex is trivial for several series of finite Coxeter groups, verifying that it is indeed a valid 3-skeletal approximation.

Due to a diagrammatic interpretation of π_2 which can be found in a book by Fenn [5] one can think of the elements of π_2 of the Salvetti and the cell complexes introduced in [4] as special types of decorated planar graphs, which we refer to as *diagrams*. Two diagrams are considered homotopic if one can be transformed into the other using a sequence of allowed transformations, which we describe in Section 2 of our paper. The problem we are considering naturally splits into two directions. One is studying unoriented diagrams, which corresponds to Coxeter groups and the topology of their classifying spaces, and the other is studying similar diagrams but with orientations on the edges, which corresponds to braid groups and the topology of the Salvetti complex. The goal of our project was to prove that any diagram is homotopic to the empty diagram in both cases (Conjecture 1), which is equivalent to the triviality of π_2 of the corresponding complexes (more details can be found in [4]). The beauty of this diagrammatic approach is the elementary nature of our combinatorial methods, which are used to prove deep statements in modern algebraic topology.

In this paper, we present our results for the symmetric groups, dihedral groups and the hyperoctahedral groups. We also present our results for the Artin braid group $\mathfrak{B}I_n$, which is a generalization of the dihedral Coxeter group. Our diagrammatic proof of the $K(\pi, 1)$ conjecturette for the aforementioned family of braid groups and our results on Coxeter groups answer a question posed in [4] regarding the existence of diagrammatic proofs for these type of statements. Our findings represent research towards proving the $K(\pi, 1)$ conjecturette for all Coxeter groups diagrammatically. In addition, our proof for the braid group $\mathfrak{B}I_m$ is among the first proofs, to our knowledge, using the diagrammatic calculus for braid groups developed in [4]. Our paper is organized as follows: In the background section we introduce all of the definitions needed for our work. In Section 3 we state several general theorems and lemmas, and as a consequence derive our result for the dihedral groups. In Sections 4, 5 and 6 we prove Conjecture 1 for the aforementioned families of Coxeter and braid groups. We conclude our work in Section 7.

2. Background

2.1. Coxeter Groups.

We begin by introducing some basic definitions associated with the study of Coxeter groups.

Definition 1 (Coxeter Group). *A Coxeter group is a group given by generators $g_1, \dots, g_n$ with relations $(g_i g_j)^{m_{i,j}} = 1$ for each pair of generators g_i, g_j, such that $m_{i,j} \in \mathbb{N}$ where $m_{i,j} \geq 2$ for $i \neq j$ and $m_{i,i} = 1$ for all i.*

Remark. *Some numbers $m_{i,j}$ can be ∞, in which case there is no relation between generators g_i and g_j. The condition $m_{i,i} = 1$ implies $g_i^2 = 1$ and as a consequence $m_{i,j} = m_{j,i}$.*

Example 2.1. *The symmetric group S_n has generators $g_1, g_2, \dots, g_{n-1}$, where g_i is i^{th} elementary transposition that sends $i \to i+1$ and $i+1 \to i$, subject to the following relations: $g_i^2 = 1$, $(g_i g_{i+1})^3 = 1$, and $(g_i g_j)^2 = 1$ if $j \neq i+1, i-1$.*

To each Coxeter group, we can associate a Coxeter-Dynkin diagram, which are a convenient method of visualizing the generators and relations of a Coxeter group and also useful in classifying the Coxeter groups.

Definition 2. *For a particular Coxeter group, the associated Coxeter-Dynkin diagram is a graph where vertices correspond to generators of the group, and edges correspond* *to relations between generators. We let v_i and v_j be arbitrary vertices in our diagram corresponding to the generators g_i and g_j respectively. We have the following 3 properties:*

- *If there is no edge between v_i and v_j, then g_i and g_j commute (i.e. $m_{i,j} = 2$).*
- *If there is an unlabelled edge between v_i and v_j, then $m_{i,j} = 3$.*
- *If there is an edge labeled with an integer k, for some $k \in \mathbb{N}$, between v_i and v_j then $m_{i,j} = k$.*

Example 2.2. *The symmetric group S_n has the following Coxeter-Dynkin diagram with $n - 1$ vertices:*

FIGURE 1. The Coxeter-Dynkin diagram of the symmetric group.

Irreducible Coxeter groups are Coxeter groups that have connected Coxeter-Dynkin diagrams. All Coxeter groups are the direct product of irreducible Coxeter groups. In our research, we only need to consider the irreducible Coxeter groups due to Theorem 3.1. Using Coxeter-Dynkin diagrams, one can classify all finite, irreducible Coxeter groups [7]. We note that the Coxeter group A_n is isomorphic to the symmetric group S_n and the Coxeter group I_n is isomorphic to the dihedral group D_n. For the rest of this paper, we use A_n and I_n in place of S_n and D_n respectively.

2.2. **Diagrammatics of Coxeter groups.** Given a Coxeter group W we assign each generator a unique color and color the vertices of the Coxeter-Dynkin diagram by the color of the corresponding generator. Having assigned each generator a color, we consider certain diagrams, which are colored planar graphs along with a number of distinct circles (edges with no attached vertices). Every edge of the diagram is colored by a color corresponding to one of the generators of the Coxeter group. Every vertex of the diagram must correspond to a pair of generators g_i and g_j, and must have degree $2m_{i,j}$ with edges alternating between the colors of the two generators. Note that there can be many vertices corresponding to the same pair of generators.

Two diagrams are considered homotopic if one can be transformed into another through a series of the following transformations:

Circle relation: Given a graph, we can add or remove empty circles of any color.

Bridge relation: Given two edges of the same color, we can switch around the vertices they connect to as long as we do not create any new intersections.

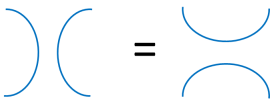

FIGURE 2. A drawing of the bridge relation.

Remark. *Note that the circle and bridge relation allow us to remove all circles in our graph. Thus we assume we have no circles in our diagrams from this point forth.*

Cancellation of pairs relation: For any subgroup of a Coxeter group G isomorphic to I_m we have the following allowed transformation. Note that that we draw the relation for I_3, but the relation holds for general I_m.

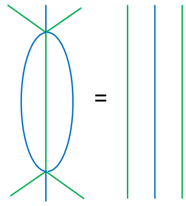

Figure 3. The cancellation of pairs relation corresponding to I_3. In general, the cancellation of pairs relation for I_m looks similar to the above picture, however each vertex has degree $2m$.

Finally, we have a class of relations called the *Zamolodchikov* relations (written as ZAM relations for brevity) that are determined by inspection of the reduced expression graph for the longest element of the finite rank 3 Coxeter groups: A_3, B_3, H_3 and $A_1 \times I_n$ (more detail can be found in [4]). All of the ZAM relations we use in our paper are drawn below. Note that while we draw the ZAM relations below with specific colors, they hold true for any colors that form the arrangement of vertices and edges in the drawings below.

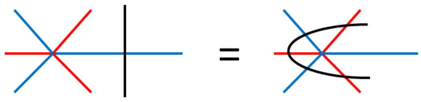

Figure 4. The ZAM relation corresponding to the group $A_1 \times A_2$.

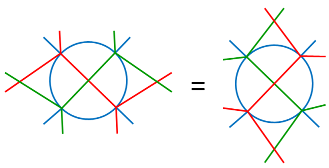

Figure 5. The ZAM relation corresponding to the group A_3.

2.3. **Braid groups and their diagrammatics.** We also consider the diagrammatics of Artin braid groups, which are a generalization of Coxeter groups. The Artin braid groups are far more complicated to work with diagrammatically, as our diagrams become oriented planar graphs with a number of oriented circles (as we will describe below).

Definition 3. *An Artin braid group is given by generators $g_1, \ldots, g_n$ with relations $(g_i g_j)^{m_{i,j}} = 1$ between every pair of generators g_i, g_j with $i \neq j$ such that $m_{i,j} \in \mathbb{N}$ and $m_{i,j} \geq 2$. There is no longer any relation $m_{i,i}$ as there was in the Coxeter groups.*

In this paper we only work with the braid group $\mathfrak{B}I_m$, which is defined below.

Definition 4. *The braid group $\mathfrak{B}I_m$ is given by 2 generators g_1 and g_2 with $m_{1,2} = m$. For the rest of our paper, when considering g_1 and g_2 as part of $\mathfrak{B}I_m$, we assign g_1 the color blue and g_2 the color green.*

Similar to how we constructed diagrams for Coxeter groups, we do the same for the braid groups as follows. For a given braid group $\mathfrak{B}$ with generators $g_1, g_2, \ldots, g_n$ and with relations $(g_i g_j)^{m_{i,j}} = 1$ between generators, we assign a distinct color to each generator. Every edge of the graph must be a color corresponding to a generator. Every vertex of the graph must correspond to a pair of generators, g_i and g_j, and must have degree $2m_{i,j}$ with edges alternating between the colors of the two generators. In addition, our edges have orientations as specified: each vertex must have $m_{i,j}$ consecutive edges pointing out of the vertex and $m_{i,j}$ consecutive edges pointing towards the vertex. One can see that orienting edges in this manner results in two distinct types of vertices for each pair of generators.

Example 2.3. *For the braid group $\mathfrak{B}I_3$, there are 2 different types of vertices, drawn below in Figure 6. Recall that we have assigned g_1 the color blue and g_2 the color green.*

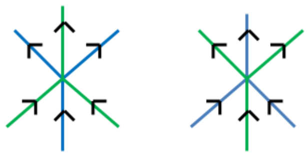

FIGURE 6. The 2 different types of vertices in diagrams for the braid group $\mathfrak{B}I_3$.

Again, two diagrams are considered homotopic if one can be transformed into the other through the following 3 transformations. Note that these are the same transformations as for the diagrammatics of Coxeter groups with added orientations.

Circle relation: Given a graph, we can add or remove empty oriented circles of any color.

We note that as for the Coxeter circle relation, this relation allows us to ignore any oriented circles in our diagrams, as we can simply remove them.

Directed bridge relation: Given two edges of the same color, we can switch around the vertices they connect to as long as we do not create any new intersections and the orientations of the edges are preserved.

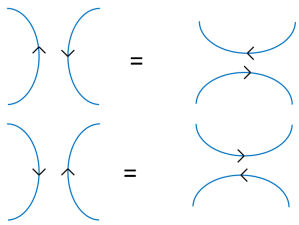

FIGURE 7. A drawing of the directed bridge relation.

Directed cancellation of pairs: For any subgroup of a braid group $\mathfrak{B}$ isomorphic to $\mathfrak{B}I_m$ we have the following allowed transformation. Note that that we draw the relation for the group $\mathfrak{B}I_3$ but it holds for general $\mathfrak{B}I_m$.

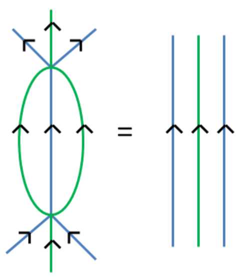

FIGURE 8. The cancellation of pairs relation corresponding to the group $\mathfrak{B}I_3$.

In general, the cancellation of pairs relation for the group $\mathfrak{B}I_m$ looks similar to Figure 8, however each vertex has degree $2m$. This cancellation of pairs relation also holds if all arrows in the above picture are reversed in orientation (i.e. they all point downwards). The braid groups also have ZAM relations corresponding to rank 3 subgroups, however we will not need any in our proofs and thus do not include them in this paper.

2.4. **The $K(\pi, 1)$ conjecturette.** The goal of our project was to prove the following conjecture.

Conjecture 1 ($K(\pi, 1)$ conjecturette)**.** *All diagrams for a particular Coxeter group or braid group are homotopic to the empty diagram through a series of the aforementioned allowed transformations.*

Remark. *As mentioned in our introduction, Conjecture 1 for braid groups is equivalent to showing that π_2 of the Salvetti complex of the corresponding Coxeter group is trivial, which is a part of the $K(\pi, 1)$ conjecture in representation theory. Conjecture 1 for Coxeter groups has a similar topological interpretation (see [4]), which is equivalent to showing that π_2 of the cell complex introduced in [4] is trivial, verifying that the cell complex is a valid 3-skeletal approximation for the classifying space of a Coxeter group W. We remark that Conjecture 1 for Coxeter groups is not part of the $K(\pi, 1)$ conjecture, as the $K(\pi, 1)$ conjecture deals with the topology of Salvetti complexes and Conjecture 1 for Coxeter groups deals with the cell complex introduced in [4]. However, for the purposes of this paper, we refer to Conjecture 1 for* both *Coxeter and braid groups as the $K(\pi, 1)$ conjecturette. In [4], authors Elias and Williamson raised the question of the existence of an elementary diagrammatic proof for Conjecture 1 for both Coxeter and braid groups. The results in our paper address this question by providing diagrammatic proofs for Conjecture 1 for several families of Coxeter groups and a family of braid groups.*

3. General Statements

In this section we provide several general statements that are useful in many of our proofs.

Theorem 3.1. *Given Coxeter groups G and H, if Conjecture 1 holds for G and for H, then it holds for the group $G \times H$.*

Remark. *Due to the above theorem, we see that if we prove Conjecture 1 for all irreducible Coxeter groups, then we have proven it for all Coxeter groups, as all Coxeter groups are the direct product of the irreducible Coxeter groups. This allows to consider only the irreducible Coxeter groups in our research. We note that the above theorem also holds if G and H are braid groups. The proof when G and H are braid groups is the same as the proof for when G and H are Coxeter groups, except it uses the braid version of the ZAM relation for $A_2 \times I_m$.*

Proof. We begin by observing that any generator $g \in G$ commutes with any generator $h \in H$. Given a diagram, consider the subgraph with only edges corresponding to generators of G. We call this subgraph the *g-subgraph* of the diagram. Examining any 2 adjacent vertices on the g-subgraph connected by an edge E, we see that edge E may be intersected by edges corresponding to generators $h \in H$. Thus using the ZAM relation corresponding to $A_2 \times I_m$ (Figure 4), since any pair of generators g, h with $g \in G$ and $h \in H$ commute, we can remove all edges intersecting E. By continuing this process on all edges of the g-subgraph, we can remove all edges corresponding to generators in H from the g-subgraph. Thus we end up with 2 disjoint graphs—one that has only edges corresponding to generators in G and the other that has only edges corresponding to generators in H. Since we know all diagrams for groups G and H are homotopic to the empty graph, we can reduce these 2 graphs to the empty graph. Thus, all possible diagrams of $G \times H$ are homotopic to the empty one. □

Lemma 3.1. *If two vertices of the same type (vertices that have edges of the same two colors) are connected by an edge, then we can delete both the vertices.*

Proof. Consider two connected vertices of the same type where each vertex has degree $2m$. Since these two vertices are connected by an edge, we can use the bridge relation locally to connect the other edges of these two vertices. Using the cancellation of pairs relation for I_m (Figure 3), we can remove both of these vertices. □

Below is a picture showing this process for the Coxeter group I_3 (which is equivalently A_2, the symmetric group of order 2).

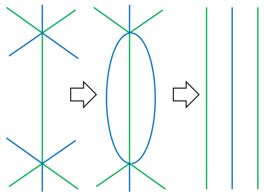

FIGURE 9. Deleting 2 adjacent vertices of the same type in the Coxeter group I_3. The first step comes from using the bridge relation. The second step comes from using the I_3 cancellation of pairs relation.

Corollary 3.1. *Conjecture 1 is true for all Coxeter groups I_n. The Coxeter groups I_n are isomorphic to the dihedral groups.*

Proof. Given the Coxeter group I_n, we see from its definition that there are only 2 generators. Thus, there is only 1 type of vertex we can form, and thus all possible diagrams will always have connected vertices of the same type. Using Lemma 3.1, we can delete these adjacent vertices, until there are no vertices left in our diagram and it is the empty diagram. □

Before introducing the next 2 lemmas, we define *boundary* and *subdiagram*—two terms that help us study local properties of diagrams.

Definition 5. *A* subdiagram *of a diagram is a subset of vertices and edges of the diagram that are connected.*

Definition 6. *Given a subdiagram, we call edges that have exactly 1 endpoint in our subdiagram and the other endpoint a vertex not in our subdiagram the* boundary *of our subdiagram.*

We note that the entire diagram has an empty boundary.

Example 3.1. *Figure 10 provides an example of a boundary.*

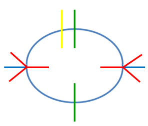

FIGURE 10. This subdiagram has boundary yellow-green-red-blue-red-green-red-blue-red.

Definition 7. *Given 2 subdiagrams D_1 and D_2 with the same boundary, we let $D_1 \sqcup D_2$ denote the diagram obtained by connecting all edges on the boundary of D_1 to the corresponding edges on the boundary of D_2.*

Definition 8. *2 subdiagrams are referred to as* equivalent *if they are homotopic.*

Lemma 3.2. *If D_1 and D_2 are two subdiagrams with the same boundary and the diagram $D_1 \sqcup D_2$ is homotopic to the empty diagram (denoted as $\varnothing$), then D_1 is homotopic to D_2.*

Proof. Clearly D_1 is equal to the diagram with D_1 and the empty graph next to it. Since $D_1 \sqcup D_2 = \varnothing$, we see that D_1 is equivalent to D_1 with the closed diagram $D_1 \sqcup D_2$ next to it. Using the bridge relation on this diagram to connect all the edges belonging to D_1 in $D_1 \sqcup D_2$ to D_1, we see that we are left with D_2 and $D_1 \sqcup D_1$. But $D_1 \sqcup D_1 = \varnothing$ since all vertices that are connected to each other are of the same type, and thus we can delete them. Thus we are left with D_2. Therefore we conclude that D_1 is homotopic to D_2. □

Corollary 3.2. *If Conjecture 1 holds for a Coxeter group W then any 2 subdiagrams D_1 and D_2 for W with the same boundary are equivalent.*

Proof. Since D_1 and D_2 have the same boundary, $D_1 \sqcup D_2$ is a closed diagram and thus is homotopic to the empty graph, since we know all diagrams for W satisfy Conjecture 1. Thus by Lemma 3.2, we see that these 2 subdiagrams are equivalent. □

Lemma 3.3. *When the boundary of a subdiagram is written as a word of the Coxeter group, the word is equivalent to the trivial word.*

Proof. Each edge in a diagram represents a specific generator (as we color the edges for this purpose). Thus, the boundary of a subdiagram represents a word formed by the generators of the group. Also, we see that every vertex in our diagram represents an equivalent rewriting of a word, because of our group relations. Thus the word formed by one part of the boundary region is transformed, and thereby, equivalent to the word formed by the other part of the boundary region through all vertices in the diagram. However, letting one of the parts of the boundary be the empty region of the boundary, which corresponds to the trivial word, we see that the word formed by the entire boundary must be equivalent to the trivial word.

□

4. Conjecture 1 for A_n

We start by fixing the following Coxeter-Dynkin diagram for A_n for the rest of our paper.

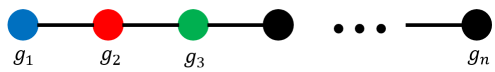

Figure 11. A colored Coxeter-Dynkin diagram for A_n. We note that in the above figure and throughout the rest of this paper, black will denote an arbitrary color that has not yet been assigned to a generator.

Lemma 4.1. *Let the Coxeter group A_n have generators $g_1, \ldots, g_n$. Every word in A_n can be written with at most 1 occurrence of g_1.*

Proof. This can be proven easily by using induction on the length of the word. □

Definition 9. *Given a pair of colors in our diagram, the two colors are said to* commute *if their corresponding generators commute.*

Definition 10. *Given c an arbitrary color, we let the c* subgraph *of a diagram denote the graph on the vertices that have adjacent edges of color c and all the c edges in which we ignore vertices of degree 2 by gluing the edges.*

Remark. *We see that all vertices of degree 4 with the color c in our original diagram are vertices with colors c/d where d is a color that commutes with c. These vertices are vertices of degree 2 in the c subgraph and are thus ignored. As a result, when considering the c subgraph, we ignore vertices formed by colors that commute with c.*

Example 4.1. *Figure 12 provides an example of a subdiagram.*

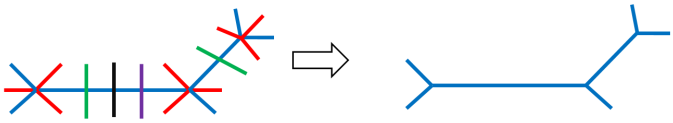

FIGURE 12. The picture on the right is the blue subgraph of the subdiagram to the left.

Definition 11. *Letting c be an arbitrary color, we let c*-face *stand for a face in the c subgraph.*

Definition 12. *Two vertices of the same type with colors c_1 and c_2 are called* almost c_1-adjacent *if they are connected by an edge E in the c_1 subgraph.*

Lemma 4.2. *Given two almost blue-adjacent vertices of type blue/red in a diagram for A_n, either they can be deleted or they can be transformed into the diagram in Figure 13.*

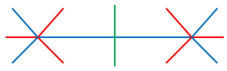

FIGURE 13. 2 almost-blue adjacent vertices of type blue/red.

Proof. Let the blue edge connecting the 2 blue/red vertices in the blue subgraph be denoted as E. We see that all the vertices on the edge E (in the context of the entire diagram and not just the blue subgraph) correspond to a word in A_n comprised of generators $g_3, \ldots, g_n$. The group generated by $g_3, \ldots, g_n$ with relations between these generators as given in the Coxeter-Dynkin diagram in Figure 11 is isomorphic to the Coxeter group A_{n-3}. Using Lemma 4.1, we can rewrite any word in A_{n-3} with at most 1 instance of g_3. Thus we can replace the vertices on the edge E with equivalent vertices, where there is at most 1 vertex of type blue/green. However, given any vertex of type blue/c where c is a color that is not green, we see that c commutes with both red and blue, and thus can be moved out of the edge E using the ZAM relation in Figure 4.

Thus, moving out all colors c that commute with red and blue, we are left with at most 1 vertex on edge E that is of type green/blue. If there are no vertices on edge E, then we have 2 adjacent vertices of the same type connected by an uninterrupted edge, and thus we can use Lemma 3.1 to remove them. In the other case, we are left with precisely the diagram in the statement of this lemma (Figure 13). □

Lemma 4.3. *The 2 diagrams in Figure 14 are equivalent.*

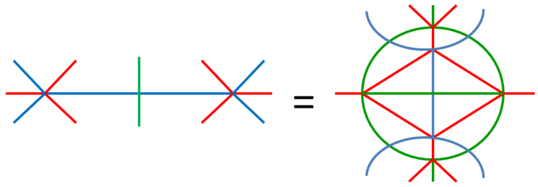

FIGURE 14. 2 equivalent diagrams.

Proof. To show that the two diagrams in Figure 14 are equivalent, we connect the boundaries of the 2 diagrams, and note that this is possible as both diagrams have the same boundary. After connecting the boundaries of the 2 diagrams, we get a closed diagram. Using the A_3 ZAM relation, we can show that the resulting diagram is equivalent to the trivial one. Using Lemma 3.2, we see that the 2 diagrams are equivalent.

□

Having stated and proven the necessary lemmas, we now provide our proof for A_n and how we solved certain challenges that arose. The fundamental idea of our proof is using induction on the number of blue/red vertices (vertices corresponding to generators g_1 and g_2). One of the major challenges we faced while trying to prove A_n was that there are many different types of vertices. Initially, we tried to use Lemma 5.1 (used in the proof for B_n later on) to find a small face and then show there must be vertices around this small face that we can delete. However, due to the many different types of vertices, this was impossible. We instead came up with Lemma 4.3, which allows us to reduce the size of a face in the blue-graph. This strategy is *very* powerful, as it allows us to consider only 1 case for a blue-face: a blue-face of size 2, from where we can find blue/red vertices to delete. Thus Lemma 4.3 is a very important tool that we have developed. This lemma is also invaluable for the case B_n, and can be applied to any Coxeter group that has a subgroup isomorphic to A_n for some $n \geq 3$.

Theorem 4.1. *Conjecture 1 holds for the family of Coxeter groups A_n.*

Remark. *The symmetric group which is one of the most common groups found throughout mathematics is a Coxeter group of type A_n.*

Proof. To prove that all diagrams for A_n are homotopic to the empty diagram, we first induct on n. We see that our base case is A_2, which is isomorphic to I_3, which we have already proven from Corollary 3.1. Now we wish to show given an arbitrary diagram for A_n that there is at least 1 blue/red vertex that we can delete. In the blue subgraph, consider a blue-face of arbitrary size. Let one vertex on the blue-face be X and an adjacent vertex be Y. Also, let the blue edge between X and Y be E. In the conext of the entire diagram (and not just the blue subgraph) X and Y are almost-blue adjacent. By Lemma 4.2, we can either delete X and Y in which case we are done, or we can transform X, Y and E to the diagram in Figure 15.

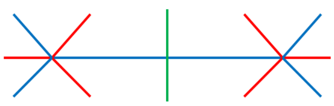

FIGURE 15. A diagram of X, Y and E.

In the case where we have a blue/green vertex on E, we use Lemma 4.3 to reduce the size of the blue-face that we have been considering. It is not obvious how Lemma 4.3 actually reduces the size of the blue-face, thus we draw an example with a blue-face of size 4 in Figure 16.

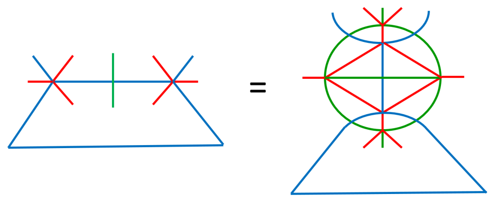

FIGURE 16. In the drawing above, we see that after applying Lemma 4.3, we reduce the size of the blue-face from 4 to 3.

We can continue this process of reducing the size of the blue-face until we are left with a blue-face of size 2. Using our method of removing colors that commute with both blue and red, there are only three possible blue-faces of size 2 where both red/blue cannot be removed, drawn below. We note that we have assigned to generator g_4 the color yellow, and see that for the group A_3, we do not have a generator g_4, and thus only one of the following blue-faces (the one with no yellow edges) is possible for the case of A_3.

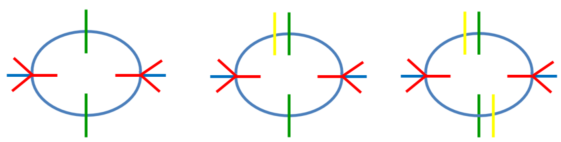

FIGURE 17. The three possible blue-faces of size 2.

Recall that red, green and yellow represent g_2, g_3 and g_4 respectively. We know from Lemma 3.3 that the boundary of any subdiagram must be equivalent to the trivial word. However, it is simple to show that the boundaries of all of the blue-faces in Figure 4 are not equivalent to the identity. Thus none of the 3 blue-faces in Figure 4 are possible, and thus the only blue-face of size 2 that is possible is drawn below:

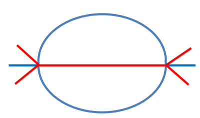

FIGURE 18. The only possible blue-face of size 2.

We note that in the above picture, we can assume the blue-face has nothing inside it except 2 connected red edges by the following reasoning: All of the colors inside the blue-face correspond to generators $g_2, \ldots, g_n$, which generate the group A_{n-1}. However, by our inductive hypothesis stated at the beginning of this proof, we can assume A_{n-1} satisfies Conjecture 1, and thus by Corollary 3.2, all subdiagrams for A_{n-1} with the same boundary are equivalent. Thus all subdiagrams of A_{n-1} that have 2 red edges are equivalent to the one where the 2 red edges are connected, as in the picture above.

Since the blue-face in Figure 18 has adjacent vertices of the same type, by Lemma 3.1, we can delete these blue/red vertices. Using induction on the number of blue/red vertices, in any diagram for A_n we can delete all blue/red vertices until there are none left. After this, all the remaining blue vertices in a diagram are blue/c, where c is a color that commutes with blue. Thus we can trivially remove all remaining blue vertices. Therefore, we have no more blue edges in our diagram, and our diagram is now a diagram for the group A_{n-1}. However now we can use our inductive hypothesis and reduce our diagram to the empty graph. Thus, all diagrams for the group A_n are homotopic to the empty graph.

□

5. Conjecture 1 for B_n

We have also proven Conjecture 1 for the Coxeter groups B_n, which has the Coxeter-Dynkin diagram drawn below:

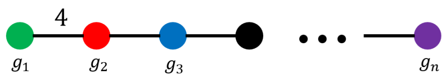

Figure 19. The Coxeter-Dynkin diagram of the group B_n.

The B_3 ZAM relation is drawn in Figure 20, and as one can see, it is much more complicated than the ZAM relations we have been using thus far. As a result, the proof for B_n is more complicated than the proof for I_m or A_n.

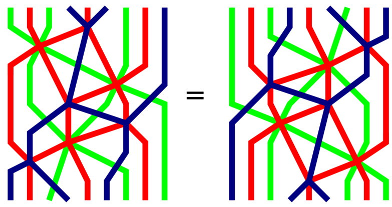

Figure 20. The B_3 ZAM relation from [4].

We first state a lemma that is a general statement about planar graphs that can be applied to our diagrams.

Lemma 5.1. *Given a planar graph where each vertex has degree 4, there is a face in the graph with size* ≤ 3.

The proof of the above lemma is standard and follows directly from the Euler characteristic for planar graphs [6].

We first prove that Conjecture 1 holds for the group B_3, which will serve as our inductive hypothesis.

Lemma 5.2. *Conjecture 1 holds for the Coxeter group* B_3.

Proof. We first use Lemma 5.1 to find a green-face of size ≤ 3 in the green subgraph of a given diagram. We then show for each possible size of such a small green-face that there are either adjacent vertices of the same type that we can delete, or we can use the B_3 ZAM relation to reorient our face in such a way that we have two adjacent vertices of the same type. In particular, the green-face clearly cannot have 1 edge, as it is impossible to have only 1 red edge entering the green face from the boundary. If the green-face has size 2 then we can delete it as previously established. Thus the only nontrivial case is where the green-face has size 3. In this scenario, to prevent there from being adjacent vertices of the same type (in which case we can delete them), we must have green/blue vertices between each green/red vertex of the face. Thus the green-face must look exactly as it does in the B_3 ZAM relation. After using the B_3 ZAM relation to reorient the green-face there must be adjacent vertices of the same type in the faces surrounding the green-face which we can delete, and thus we are done. □

Theorem 5.1. *Conjecture 1 holds for the Coxeter group B_n.*

Proof. As before, we use induction on n, assuming that B_i for $i < n$ satisfies Conjecture 1. Note that our base case B_3 was proven in Lemma 5.2. Now take an arbitrary diagram corresponding to B_n (where $n > 3$) and examine the purple subgraph (recall that purple corresponds to the generator g_n). We assign the generator g_{n-1} the color gray. We wish to show that in any such diagram, there are purple/gray vertices we can delete. We see that Lemma 4.3 applies to the purple generator and generators g_{n-1} and g_{n-2}, as these 3 generators generate a subgroup of B_n isomorphic to A_3. Thus, using Lemma 4.3 and the same procedure as for A_n, we can take an arbitrary purple-face and can obtain a purple-face of size 2.

We see that if we are unable to delete the 2 purple/gray vertices, then there must be an edge corresponding to generator g_{n-2} crossing both edges of the purple-face (by the same logic as with the A_n case). If we let the word w correspond to the edges crossing one edge of the purple-face and v correspond to the edges crossing the other edge of the purple-face, we see that the boundary of our purple-face is the word $g_{n-1}g_ng_{n-1}wg_{n-1}g_ng_{n-1}v$, where both w and v contain at least 1 occurrence of g_{n-2} and no occurrences of g_n or g_{n-1}. Using the relation $(g_{n-1}g_n)^3 = 1$ and $g_n^2 = 1$, and also the fact that g_n commutes with all generators in w and v, we can rewrite the word corresponding to our boundary as:

$$g_ng_{n-1}g_nwg_ng_{n-1}g_nv = g_ng_{n-1}wg_ng_ng_{n-1}g_nv = g_ng_{n-1}wg_{n-1}g_nv = g_ng_{n-1}wg_{n-1}vg_n.$$

The only way that we can delete both instances of g_n is if we can also delete both instances of g_{n-1}. But since w has at least 1 occurrence of g_{n-2} and no occurrences of g_{n-1}, and g_{n-2} does not commute with g_{n-1}, this is impossible. Thus any boundary of the purple-face where we cannot delete the 2 purple/gray vertices is not equivalent to the trivial word and thus is not a valid boundary by Lemma 3.3. Therefore, we can always delete the purple vertices of the purple-face of size 2, and thus in any diagram we can find 2 purple/gray vertices to delete. After we use this procedure to delete all purple/gray vertices from our diagram, the only purple vertices left in our graph are of type purple/c where c is a color that commutes with purple. Thus we can remove all remaining purple edges trivially, and are left with a graph for B_{n-1}, which we can remove using our inductive hypothesis. □

6. Conjecture 1 for $\mathfrak{B}I_n$

Finally, we prove Conjecture 1 for the dihedral Artin braid groups. While this group has only two generators, it is a very difficult case due to the orientations on our planar graphs. The main challenge we faced is that two adjacent vertices of the same type are not necessarily deletable because orientations of the vertices may not line up. Additionally, using Lemma 5.1 to find a small face did not yield any helpful method of finding vertices to delete. Our key insight in this case was to look at angles in our graph and rephrase our problem in terms of properties of these angles. These same properties of angles that we examine have been useful in our work thus far in other oriented cases.

Recall for $\mathfrak{B}I_n$ we assign g_1 the color blue and g_2 the color green, as in Definition 4.

Definition 13. *An angle of a vertex in a diagram is called* varied *if the edges of the angle have different orientations. An angle is called* uniform *if the edges of the angle have the same orientation.*

Example 6.1. *Figure 21 provides an example of varied and uniform angles.*

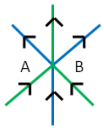

FIGURE 21. In this vertex for $\mathfrak{B}I_3$, angles A and B are varied angles. The rest of the angles are uniform angles.

Theorem 6.1. *Conjecture 1 holds for the braid groups $\mathfrak{B}I_n$.*

Proof. We note that if a face in our graph has 2 adjacent varied angles, then we can use the bridge relation to obtain a face of size 2, which we can then remove using the $\mathfrak{B}I_n$ cancellation of pairs relation. Also, we see that a face of size 2 has either 2 varied angles (in which case we can delete it) or 2 uniform angles. Having made these observations, we prove that in any given diagram, there is at least 1 vertex that we can delete.

We first let F stand for the number of faces in our graph, E stand for the number of edges and V stand for the number of vertices. Assume for the sake of contradiction that in a given diagram there are no vertices we can delete. We see that every face must have no 2 adjacent angles be varied angles. Thus all faces must have at least 2 uniform angles (including faces of size 2, since faces of size 2 have either 2 varied or 2 uniform angles, and if they have 2 varied angles the vertices of the face can be deleted using the directed cancellation of pairs relation). Thus the number of uniform angles is $\geq 2F$. However using the Euler formula for planar graphs, we know that $V + F = E + 2$, where in this case $E = \frac{2nV}{2} = nV$, since each vertex of our graph has degree $2n$. Thus $F = (n-1)V+2$. Thus the number of uniform angles is $\geq 2F = 2((n-1)V+2) = (2n-2)V+4$. But every vertex has precisely $2n-2$ uniform angles, thus the number of uniform angles is exactly $(2n-2)V$. Thus the number of uniform angles is simultaneously $\geq (2n-2)V+4$ but equal to $(2n-2)V$. Clearly, this is a contradiction, and thus these exists a face with 2 adjacent varied angles, and we have found 2 vertices we can delete (using the $\mathfrak{B}I_n$ cancellation of pairs relation). Using induction, we can delete all vertices from our graph. Therefore, every diagram for $\mathfrak{B}I_n$ is homotopic to the empty diagram.

□

7. Conclusion and Future Directions

The work presented in this paper uses the diagrammatics of Coxeter groups and braid groups to diagrammatically prove the $K(\pi, 1)$ conjecturette for several families of Coxeter groups: I_n, A_n, and B_n. Additionally, we diagrammatically prove this conjecture for the braid group $\mathfrak{B}I_m$. This work addresses a question posed in [4] regarding the existence of a diagrammatic proof of the $K(\pi, 1)$ conjecturette, which the authors of [4] were unable to find an elementary proof for. Perhaps the most important aspect of our work is the development of lemmas and strategies that can be used to tackle further cases of the $K(\pi, 1)$ conjecturette for other Coxeter groups and braid groups—especially for cases that remain unsolved with traditional approaches. In particular, Lemma 4.3 can be applied to any Coxeter group that has a subgroup isomorphic to A_n for $n \geq 3$. Additionally, our approach towards the oriented case $\mathfrak{B}I_m$ has yielded partial results for other braid groups. With this in mind, one might consider generalizing our work to the type D Coxeter groups or extending our methods to other families of braid groups.

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