

## The PRIMES 2013 Math Section problem set

Dear PRIMES applicant,

If you are applying to the Math Section, please send us your solutions of this problem set as part of your PRIMES application by December 1, 2012. For complete rules, see <http://web.mit.edu/primes/apply.shtml>

Note that this set contains two parts: “General math problems” and “Advanced math problems”. Please solve as many problems in both parts as you can. You can type the solutions or write them down by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, ”smith-solutions.” Include your full name in the heading of the file.

Please write down not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, **except** other people’s help. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day, and think about them, on and off, over some time (days and possibly weeks). We encourage you to apply if you can solve at least 50% of the problems. We note, however, that many factors will play a role in the admission decision besides your solutions of these problems.

Good luck!

### General math problems

**Problem G1.** You toss two coins trying to get two heads. Namely, you toss both coins until you get at least one head. If you get two heads at that point, you finish, and if you get just one, you keep tossing the other coin until you get a head.

(a) What's the chance that you will not finish in  $m$  tosses? (Tossing two coins simultaneously counts as one toss).

(b) Give the answer as a fraction for  $m = 8$ .

**Problem G2.** A black Bishop and a white King are placed on a  $2n$  by  $2n$  chessboard (all positions are equally likely). What is the chance that the Bishop attacks the King (i.e., the King is under check)? (Recall that the Bishop strikes along diagonals). Write the answer as a rational function of  $n$ .

**Problem G3.** A sequence  $a_n, n \geq 0$  of positive integers satisfies the recursion  $a_{n+1} = F(a_n)$ , where  $F(x)$  is the sum of the squares of all the digits of  $x$  (for example,  $F(324) = 3^2 + 2^2 + 4^2 = 29$ ). The number  $a_0$  can be arbitrary.

(a) Show that either  $a_n = 1$  starting from some  $n$ , or  $a_n$  is periodic with minimal period 8 starting from some  $n$ . What are the 8 numbers repeating periodically in the second case?

(b) Let  $h$  be any nonnegative integer function on the set of digits  $0, 1, \dots, 9$ . Consider the recursion  $b_{n+1} = F_h(b_n)$ , where  $F_h(x) := \sum_i h(x_i)$ , and  $x_i$  are the digits of  $x$ . Show that  $b_n$  is eventually periodic, and there are finitely many possible minimal periods for it.

**Problem G4.** Let  $a > 1$ .

(a) For which  $a$  does there exist a solution  $0 \leq x \leq 1$  of the equation

$$(1+x)^{1/3} + (1-x)^{1/3} = a?$$

Show that if such a solution exists, it is unique, and find it.

(b) For which  $a$  does there exist a solution  $0 \leq x \leq 1$  of the equation

$$(1+x)^{1/4} + (1-x)^{1/4} = a?$$

Show that if such a solution exists, it is unique, and find it.

**Problem G5.** Find all rational numbers  $a, b$  for which the number  $\gamma = \log_2(a + b\sqrt{2})$  is rational.

### Advanced math problems

**Problem M1.** You toss a coin  $n$  times. What is the probability that the number of heads you'll get is divisible by 3? (Find an exact formula, not involving sums of unbounded length; it may depend on the remainder of  $n$  modulo 6).

**Problem M2.** (a) Let  $c < 2\pi$  be a positive real number. Show that there are infinitely many integers  $n$  such that the equation

$$x^2 + y^2 + z^2 = n$$

has at least  $c\sqrt{n}$  integer solutions.

(b) Find a constant  $C > 0$  such that there are infinitely many  $n$  for which the equation

$$x^5 + y^3 + z^2 = n$$

has  $\geq Cn^{1/30}$  nonnegative solutions.

**Problem M3.** A finite string of 0s and 1s is called **admissible** if it occurs in one of the rows of the Pascal triangle modulo 2. I.e., a string is admissible if it has the form

$$\binom{n}{m}, \binom{n}{m+1}, \dots, \binom{n}{m+k},$$

where the binomial coefficients are taken modulo 2.

(a) Which strings of length  $\leq 4$  are not admissible? Why?

(b) Give an explicit description of all admissible strings of length  $n$ .

(c) What is the number  $a(n)$  of admissible strings of length  $n$ ? (Write a recursion, guess the answer and prove it by induction; the recursive formula may be different for  $n = 2r$  and  $n = 2r + 1$ ).

**Problem M4.** Positive solutions of the equation  $x \sin(x) = 1$  form an increasing sequence  $x_n, n \geq 1$ .

(a) Find the limit

$$c_1 = \lim_{n \rightarrow \infty} n(x_{2n+1} - 2\pi n).$$

(b) Find the limit

$$c_2 = \lim_{n \rightarrow \infty} n^3(x_{2n+1} - 2\pi n - \frac{c_1}{n}).$$

**Problem M5.** Let us say that a polynomial  $f$  with complex coefficients is **degenerate** if there exists a square matrix  $B$  such that  $B \neq f(A)$  for any square matrix  $A$ . What are all the degenerate polynomials of degree 2? degree 3? degree 4? any degree?