# Equivalence classes of length-changing replacements of size-3 patterns

Vahid Fazel-Rezai
Mentor: Tanya Khovanova

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# Outline

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1. Definitions
2. Results
   - $\beta$ Decreasing
   - Shift Right, Shift Left
   - Drop Only
   - Drop One, Swap $*$ with Neighbor

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# Permutations and Patterns

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Special permutations:

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## Definition

Let $p$ be a string of distinct positive integers. A substring of a permutation $\pi$ order-isomorphic to $p$ is a **copy** of the **pattern** $p$ in $\pi$. If no such substrings exist, $\pi$ **avoids** $p$.

# Replacements

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Let $\alpha$ and $\beta$ be strings, of equal length, of distinct integers and $*$. Then, $\sigma$ is the result of a **replacement** $\alpha \to \beta$ on $\pi$ if $\sigma$ is obtained by:

1. adding instances of $*$ in $\pi$ as necessary,
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# Equivalence

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Two permutations $\pi$ and $\sigma$ are **equivalent** ($\pi \equiv \sigma$) under $\alpha \leftrightarrow \beta$ if $\sigma$ can be attained through a sequence of $\alpha \to \beta$ or $\beta \to \alpha$ replacements on $\pi$.

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Example: Under $123 \leftrightarrow 3{*}2$, we have

$$14253 \equiv 4312.$$

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    $123 \leftrightarrow {*}12$ and $123 \leftrightarrow 23{*}$
  - Drop Only (3 cases)
    e.g. $123 \leftrightarrow 12{*}$
  - Drop, Swap $*$ with Neighbor (4 cases)
    e.g. $123 \leftrightarrow 1{*}2$

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# Two Lemmas

**Lemma**

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**Lemma**

*If $\beta$ is decreasing, all identity permutations of length 4 or greater are equivalent.*

# Finitely Many Classes

**Theorem**

*If $\beta$ is decreasing, there are five equivalence classes:*

$$\{\emptyset\}, \{1\}, \{12\}, \{123, 21\}, \{\textit{everything else}\}$$

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3. Future Plans

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We observe the following:

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**Theorem**

*Under* $123 \leftrightarrow *12$ *and* $123 \leftrightarrow 23*$*, each reverse identity is in a distinct class while all other permutations are equivalent.*

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# Shortest Equivalent Permutation

**Lemma**

*Apply the replacement* $123 \to \beta$ *as many times as possible (in any order) to some* $\pi$*, and call the result* $p(\pi)$.

- $p(\pi)$ *is the unique shortest permutation equivalent to* $\pi$.
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**Lemma**

*Apply the replacement 123 $\to \beta$ as many times as possible (in any order) to some $\pi$, and call the result $p(\pi)$.*

- *$p(\pi)$ is the unique shortest permutation equivalent to $\pi$.*
- *$p(\pi)$ avoids 123.*

Thus, there is a bijection between equivalence classes and permutations avoiding 123:

**Theorem**

*Under drop only replacements, for each $\sigma$ avoiding 123, there exists a distinct class containing all $\pi$ with $p(\pi) = \sigma$.*

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# Alternative Equivalence

The following lemma allows previous work to be applied here.

**Lemma**

*There exists some length-preserving replacement under which equivalence implies equivalence under* $123 \to \beta$ *for each* $123 \to \beta$ *in this category.*

For example, equivalence under $123 \leftrightarrow 132$ implies equivalence under $123 \leftrightarrow 13*$.

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## Theorem

*Two permutations are equivalent under* $123 \leftrightarrow 13*$ *if and only if they have the following in common:*

- *number of left-to-right minima,*

*and out of the elements that are not left-to-right minima,*

- *leftmost position, and*
- *largest value (relative to left-right minima).*

*The other three replacements have similar invariants.*

# Future Work

I plan to continue this research by:

- characterizing equivalence classes of $132 \leftrightarrow \beta$ replacements
- considering the case when $\beta$ contains two $*$
- generalizing to longer patterns
- exploring the shortest distance between two permutations
- examining why some replacements have the same classes

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Questions?