ORBITS OF A FIXED-POINT SUBGROUP OF THE SYMPLECTIC GROUP ON PARTIAL FLAG VARIETIES OF TYPE A

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Abstract. In this paper we compute the orbits of the symplectic group Sp_{2n} on partial flag varieties GL_{2n}/P and on partial flag varieties enhanced by a vector space, $\mathbb{C}^{2n} \times \mathrm{GL}_{2n}/P$. This extends analogous results proved by Matsuki on full flags. The general technique used in this paper is to take the orbits in the full flag case and determine which orbits remain distinct when the full flag variety GL_{2n}/B is projected down to the partial flag variety GL_{2n}/P.

The recent discovery of a connection between abstract algebra and the classical combinatorial Robinson-Schensted (RS) correspondence has sparked research on related algebraic structures and relationships to new combinatorial bijections, such as the Robinson-Schensted-Knuth (RSK) correspondence, the "mirabolic" RSK correspondence, and the "exotic" RS correspondence. We conjecture an exotic RSK correspondence between the orbits described in this paper and semistandard bi-tableaux, which would yield an extension to the exotic RS correspondence found in a paper of Henderson and Trapa.

1. Introduction

1.1. **Background.** Group actions are fundamental to representation theory. The group action of a group G on a set X is a homomorphism from G to the symmetric group of X, i.e. any element of G corresponds to a permutation of the elements of X. An important property of a group action is the *orbits* into which the group action of G divides X. Two elements $x_1, x_2 \in X$ are in the same orbit iff there exists some $g \in G$ such that $gx_1 = x_2$. The relation of x_1, x_2 being in the same orbit is an equivalence relation; thus, the orbits of G on X partition X into finitely or infinitely many orbits. In this paper we classify orbits of Sp_{2n} on GL_{2n}/P and orbits of a fixed-point subgroup of Sp_{2n} on GL_{2n}/P, where Sp_{2n} denotes the symplectic group and GL_{2n}/P denotes a partial flag variety of type A.

A major motivation for the results of this paper is the classical Robinson-Schensted (RS) correspondence, which is a combinatorial bijection between the symmetric group S_n (permutations on n elements) and pairs of standard Young tableaux on n boxes of the same shape. This was discovered independently in different forms by Robinson in 1938 (see [2]) and Schensted in 1961 (see [7]).

Moreover, the Robinson-Schensted correspondence runs much deeper than a simple combinatorial bijection. The Bruhat decomposition of a connected reductive group G is the decomposition of G into double cosets BwB where B is a Borel subgroup of G, W is the Weyl group of G, and $w \in W$. Alternatively, the orbits of B on G/B are in bijection with W. In the case where G is GL_n, $W = S_n$ and B can be taken as the subgroup of invertible upper triangular matrices; then the Bruhat decomposition states that every $g \in \mathrm{GL}_n$ can be written as $g = b_1 w b_2$ where b_1, b_2 are invertible and upper triangular and w is a permutation matrix. In this way the RS correspondence has a natural bijection with the orbits of B on GL_n/B.

The Robinson-Schensted-Knuth (RSK) correspondence is a generalization of the RS correspondence. It extends the RS correspondence to a bijection between non-negative integer matrices with entries summing to n and pairs of semistandard Young tableaux on n boxes of the same shape, where the column sums of the matrix must equal the weight of the first tableau, and the row sums of the matrix must equal the weight of the second tableau. Note that this reduces to the standard RS correspondence when the matrices are permutation matrices, corresponding to S_n, and the semistandard Young tableaux are standard Young tableaux, corresponding to weights of $(1, 1, ..., 1)$. Whereas the RS correspondence parameterizes B-orbits on GL_n/B, the RSK correspondence parameterizes P-orbits on GL_n/P, where P is a parabolic subgroup of GL_n.

Spaltenstein discovered that one can recover the RS correspondence by classifying the irreducible components of the Steinberg variety. The Steinberg variety is a certain subset of the product of two copies of the flag variety with the cone of nilpotent elements of the Lie algebra $\mathfrak{sl}_n$, and its irreducible components can be naturally parametrized in two different ways. A concise treatment is given in Chapter 3 of [1]. Furthermore, this geometric correspondence was generalized by Rosso (2010, see [6]) to the partial flag variety to recover the full RSK correspondence.

Travkin (2011, see [8]) computed the orbits of B on $\mathbb{C}^{2n} \times \mathrm{GL}_{2n}/B$ and from this found the mirabolic RSK correspondence, a bijection between decorated permutations (w, β) where $w \in S_n$ and β is a subset of $\{1, ..., n\}$ restricted by w, and triples (T_1, T_2, θ) consisting of a pair of standard Young tableaux and an extra partition restricted by T_1 and T_2.

Matsuki (2010, see [5]) gave a characterization of the orbits of the symplectic group Sp_{2n} on $\mathbb{C}^{2n} \times \mathrm{GL}_{2n}/B$, where GL_{2n}/B is the complete flag variety in $\mathbb{C}^{2n}$. Henderson and Trapa (2011, see [3]) use this to find an exotic version of the RS correspondence. They give a bijection between pairs (w, β) as in mirabolic RSK but with w restricted to fixed point-free involutions, and standard Young bi-tableaux for certain bi-partitions.

A major inspiration of this work is Rosso's program [6] to extend Robinson-Schensted-type correspondences using partial flags. Hence we would like to generalize the exotic Robinson-Schensted above to one involving semistandard bi-tableaux by extending the work of Matsuki and Henderson and Trapa to partial flags.

A special case of one of our results is contained in Magyar (see [4]) using the theory of quiver representations. The results of this paper can be applied to studying certain categories of representations.

1.2. **Main results.** In Section 2 we lay out the preliminary definitions and concepts and state several results of Matsuki, which are heavily used throughout the rest of the paper. In Section 3 we compute Sp_{2n}-orbits on GL_{2n}/P, the main result being Theorem 3.10. Finally, in Section 4, we compute Q-orbits on GL_{2n}/P (where Q is a fixed-point subgroup of Sp_{2n}), the main result being Theorem 4.6, and the orbits of Sp_{2n} on $\mathbb{C}^{2n} \times \mathrm{GL}_{2n}/P$ obtained as a corollary (Corollary 4.7).

1.3. **Acknowledgements.** I would like to thank my mentors Vinoth Nandakumar and Daniel Thompson for teaching me background and guiding me through my research; Ben Elias, Pavel Etingof, and Slava Gerovitch for supervising the MIT-PRIMES program; and my parents for their continued support and guidance.

2. Preliminaries

2.1. **Basics.** The vector space $V = V_n = \mathbb{C}^n$ is the n-dimensional complex vector space. The *general linear group* GL_n, referred to as G, is the group of invertible linear transformations on $\mathbb{C}^n$. The span of a set of vectors $\{v_1, ..., v_k\}$ is denoted $\langle v_1, ..., v_k\rangle$. A *complete flag* is a sequence of subspaces $\{0 \subset V_1 \subset ... \subset V_n\}$ such that $\dim(V_i) = i$ for each i. A *complete flag variety* is the space of all complete flags. For a fixed basis, the *Borel subgroup* $B \subset G$ is the group of all invertible linear transformations leaving a chosen complete flag fixed. The complete flag variety is isomorphic to G/B.

A *partial flag* of *signature* $(d_1, ..., d_k)$, $d_1 > 0$, $d_k = n$, is a sequence of subspaces $\{0 \subset V_{d_1} \subset ... \subset V_{d_k}\}$, $\dim(V_i) = i$ for each i. Sometimes we refer to d_0, which is 0. A *partial flag variety* of signature $(d_1, ..., d_k)$ is the space of all partial flags with that signature. For a fixed basis, a *parabolic subgroup* P of signature $(d_1, ..., d_k)$ is the group of all invertible linear transformations leaving a chosen partial flag fixed. Thus $B \subseteq P$ for all parabolics P. The partial flag variety of signature $(d_1, ..., d_k)$ is isomorphic to G/P.

The *relative position* of two flags $F = \{V_1 \subset ... \subset V_n\}$ and $F' = \{V_1' \subset ... \subset V_n'\}$ is defined as the matrix $\{\dim(V_i \cap V_j')\}_{i,j=0}^n$. The action of G on a pair of flags leaves their relative position fixed, since invertible matrices do not change the dimensions of subspaces.

From here V will be assumed to be even-dimensional, and its dimension will be denoted $2n$ instead of n.

The *symplectic form* $\omega : V \times V \to \mathbb{F}$ is a bilinear form on V that satisfies three properties:

- *Skew-symmetric*: $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$.
- *Totally isotropic*: $\omega(v, v) = 0$ for all $v \in V$.
- *Nondegenerate*: For all nonzero $u \in V$, $\omega(u, v) \neq 0$ for some $v \in V$.

Two vectors u, v are called a *symplectic pair* if $\omega(u, v) \neq 0$. Otherwise, they are called *perpendicular*, and we write $u \perp v$.

For a subspace $U \subseteq V$, its perpendicular $U^{\perp} \subseteq V$ is defined as $U^{\perp} : v \in V \mid u \perp v$ for all $u \in U$. For a flag $F = \{0 \subset V_{d_1} \subset ... \subset V_{d_k}\}$, its perpendicular is defined as $F^{\perp} = \{0 \subset V_{d_{k-1}}^{\perp} \subset ... \subset V_{d_1}^{\perp} \subset V\}$.

It is possible to find a symplectic basis on all even-dimensional vector spaces V; we find such a basis $(e_1, ..., e_{2n})$ such that $\omega(e_i, e_{2n+1-i}) = 1$ for $i \leq n$, and we call it the standard basis for V. The symplectic group Sp_{2n}, referred to as K, is the group of invertible linear transformations that preserve the symplectic form, that is, the set of all $k \in \mathrm{GL}_{2n}$ such that $\omega(u, v) = \omega(ku, kv)$ for all $u, v \in V$.

Define $Q_{2n} = Q$ as the subgroup of K consisting of symplectic vectors leaving e_{2n} fixed, i.e. $Q = \{q \in K \mid qe_{2n} = e_{2n}\}$.

Define W as $\langle e_{2n} \rangle^{\perp} = \{e_2, e_3, ..., e_{2n}\}$.

Often we will refer to the relative position of a flag F with its perpendicular flag $F^{\perp}$, i.e. $\{\dim(V_i \cap V_j^{\perp})\}_{i,j=0}^{2n}$. We may refer to this simply as the relative position of F.

2.2. **Prior results.** We adopt Matsuki's notation (see [5]) $\{d_{i,j}\}_{i,j=0}^{2n}$ for the relative position of a full flag (with its perpendicular), and the derived matrix $\{c_{i,j}\}_{i,j=1}^{2n}$ defined by $c_{i,j} = d_{i,j-1} - d_{i,j} - d_{i-1,j-1} + d_{i-1,j}$.

Definition 2.1. *Define C_{2n} as the set of $2n$ by $2n$ symmetric permutation matrices with zeros along the main diagonal.*

These matrices may be viewed as ways to fully pair $2n$ elements. They may also be called fixed point-free involutions.

The following results about K-orbits on G/B were determined by Matsuki (see [5]):

Proposition 2.2. *For every flag $F = \{0 \subset V_1 \subset ... \subset V_{2n}\}$, there exists an ordered basis $(v_1, ..., v_{2n})$ of V such that $V_i = \langle v_1, ..., v_i \rangle$ for $1 \leq i \leq 2n$, which has the additional property $\{\omega(v_i, v_j)\}_{i,j=1}^{2n} \in C_{2n}$.*

Theorem 2.3. *The orbits of K on G/B are in bijection with C_{2n}.*

The following results about Q-orbits on G/B were also determined by Matsuki (see [5]):

Proposition 2.4. *For every flag $F = \{0 \subset V_1 \subset ... \subset V_{2n}\}$, there exists an ordered basis $(v_1, ..., v_{2n})$ of V such that $V_i = \langle v_1, ..., v_i \rangle$ for $1 \leq i \leq 2n$, which has the additional properties: $\{\omega(v_i, v_j)\}_{i,j=1}^{2n} \in C_{2n}$ and $\{\omega(v_i, e_{2n})\}_{i=1}^{2n}$ comprises s ones and $2n - s$ zeros, $1 \leq s \leq n$.*

Theorem 2.5. *The orbits of Q on G/B are in bijection with $\bigsqcup_{s=1}^{n} \bigsqcup_{*} C(I_{(A)})$, where $C(I_{(A)})$ is the set of $|I_{(A)}|$ by $|I_{(A)}|$ symmetric permutation matrices $\{c_{i,j}\}_{i,j \in I_{(A)}}$ with zeros along the main diagonal, and the disjoint union $*$ is taken for all partitions $\{1, ..., 2n\} = I_{(A)} \sqcup I_{(X)} \sqcup I_{(Y)}$ such that $|I_{(X)}| = |I_{(Y)}| = s$.*

Corollary 2.6. *The orbits of Q on G/B are in bijection with sequences $Z_1 ... Z_{2n}$ of "AXY" symbols: X symbols, $X_1 ... X_s$; Y symbols, $Y_1 ... Y_s$; and pairs of A symbols, $A_{s+1}, A_{s+1}, ..., A_n, A_n$, such that:*

- $1 \leq s \leq n$.
- *The symbol X_i appears before the symbol X_{i+1} for all $1 \leq i \leq s - 1$.*
- *The symbol Y_i appears before the symbol Y_{i+1} for all $1 \leq i \leq s - 1$.*
- *The first A_i symbol appears before the first A_{i+1} symbol for all $s + 1 \leq i \leq n - 1$.*

Moreover, any flag in the orbit corresponding to the sequence $Z_1 ... Z_{2n}$, when represented by a basis $(v_1, ..., v_{2n})$ described in Proposition 2.4, has the following properties:

- $\omega(v_i, v_j) = 1$ *iff the "AXY" symbols Z_i and Z_j have the same index; $\omega(v_i, v_j) = 0$ otherwise.*
- $\omega(v_i, e_{2n}) = 1$ *iff "AXY" symbol Z_i is an X symbol; $\omega(v_i, e_{2n}) = 0$ otherwise.*

Proposition 2.7. *For every orbit of Q on G/B, take from Theorem 2.5 the corresponding s, partition $I_{(A)} \sqcup I_{(X)} \sqcup I_{(Y)}$ of $\{1, ..., 2n\}$, and matrix $\{c_{i,j}\}_{i,j \in I_{(A)}}$. There exist unique subsequences $i_1 < ... < i_{n-s}$ and $j_1 < ... < j_{n-s}$ of $I_{(A)}$ with $i_t < j_t$ for $t = 1, ..., n-s$. We may take the following to be the standard basis of the orbit:*

$$u_{i_1} = e_{s+1}, \ \ ..., \ \ u_{i_{n-s}} = e_n,$$

$$u_{j_1} = e_{2n-s}, \ \ ..., \ \ u_{j_{n-s}} = e_{n+1},$$

$$u_{x_1} = e_1 + e_2, \ \ u_{x_2} = e_1 + e_3, \ \ ..., \ \ u_{x_{s-1}} = e_1 + e_s, \ \ u_{x_s} = e_1,$$

$$u_{y_1} = \varepsilon_1 e_{2n-1}, \ \ u_{y_2} = \varepsilon_2 e_{2n-2}, \ \ ..., \ \ u_{y_{s-1}} = \varepsilon_{s-1} e_{2n-s+1},$$

$$u_{y_s} = \varepsilon_s (e_{2n} - e_{2n-1} - ... - e_{2n-s+1})$$

where $\varepsilon_t = 1$ if $x_t < y_t$, or -1 otherwise. The standard flag corresponding to this basis is derived in the natural way, $V_i = u_1 \oplus ... \oplus u_i$.

3. K-orbits on G/P

For a parabolic of signature $(d_1, ..., d_k)$ where $d_0 = 0$ and $d_k = 2n$, define the matrix $\{d_{i,j}\}_{i,j=0}^{k}$ (analogously to the full flag case) by

$$d_{i,j} = \dim(V_{d_i} \cap V_{d_j}^{\perp})$$

as the relative position of a partial flag with its perpendicular. (Note that d_i has a very different meaning from $d_{i,j}$.) Then let $\{c_{i,j}\}_{i,j=1}^{k}$ be the matrix defined by

$$c_{i,j} = d_{i,j-1} - d_{i,j} - d_{i-1,j-1} + d_{i-1,j}.$$

So every flag F has a $\{d_{i,j}\}$ matrix and a $\{c_{i,j}\}$ matrix associated with it. For convenience, we will call these matrices $d(F)$ and $c(F)$ respectively.

The definition of $c(F)$ relies on $d(F)$. The following lemma proves the process is reversible, giving a formula for obtaining $d(F)$ from $c(F)$.

Lemma 3.1. *For a partial flag,* $d_{i,j} = \sum_{r=1}^{i} \sum_{s=j+1}^{k} c_{r,s}$.

Proof.

$$\begin{aligned}
\sum_{r=1}^{i} \sum_{s=j+1}^{2n} c_{r,s} &= \sum_{r=1}^{i} \sum_{s=j+1}^{2n} (d_{r,s-1} - d_{r,s} - d_{r-1,s-1} + d_{r-1,s}) \\
&= \sum_{r=1}^{i} \sum_{s=j}^{2n-1} d_{r,s} - \sum_{r=1}^{i} \sum_{s=j+1}^{2n} d_{r,s} - \sum_{r=0}^{i-1} \sum_{s=j}^{2n-1} d_{r,s} + \sum_{r=0}^{i-1} \sum_{s=j+1}^{2n} d_{r,s} \\
&= \sum_{r=1}^{i} \sum_{s=j}^{2n} d_{r,s} - \sum_{r=1}^{i} \sum_{s=j+1}^{2n} d_{r,s} - \sum_{r=1}^{i-1} \sum_{s=j}^{2n} d_{r,s} + \sum_{r=1}^{i-1} \sum_{s=j+1}^{2n} d_{r,s} \\
&= \sum_{r=1}^{i} d_{r,j} - \sum_{r=1}^{i-1} d_{r,j} \\
&= d_{i,j}.
\end{aligned}$$

□

The following lemma provides a formula for obtaining $c(F_P)$ from $c(F_B)$ where $F_B \in G/B$ projects down to $F_P \in G/P$.

Lemma 3.2. *For* $F_P \in G/P$, *extend it to any full flag* $F_B \in G/B$. *Let* $c = c(F_P)$ *and* $\gamma = c(F_B)$. *Then*

$$c_{i,j} = \sum_{r=d_{i-1}+1}^{d_i} \sum_{s=d_{j-1}+1}^{d_j} \gamma_{r,s}.$$

Proof.

$$
\begin{aligned}
c_{i,j} &= (d_{i,j-1} - d_{i-1,j-1}) - (d_{i,j} - d_{i-1,j}) \\
&= \left(\sum_{r=1}^{d_i} \sum_{s=d_{j-1}+1}^{2n} \gamma_{r,s} - \sum_{r=1}^{d_{i-1}} \sum_{s=d_{j-1}+1}^{2n} \gamma_{r,s} \right) - \left(\sum_{r=1}^{d_i} \sum_{s=d_j+1}^{2n} \gamma_{r,s} - \sum_{r=1}^{d_{i-1}} \sum_{s=d_j+1}^{2n} \gamma_{r,s} \right) \\
&= \sum_{r=d_{i-1}+1}^{d_i} \sum_{s=d_{j-1}+1}^{2n} \gamma_{r,s} - \sum_{r=d_{i-1}+1}^{d_i} \sum_{d_j+1}^{2n} \gamma_{r,s} \\
&= \sum_{r=d_{i-1}+1}^{d_i} \sum_{s=d_{j-1}+1}^{d_j} \gamma_{r,s}.
\end{aligned}
$$

□

Due to this lemma, a natural way of thinking about the $c_{i,j}$ for partial flags is as the number of symplectic pairs between two partitions, or more rigorously, the number of symplectic pairs between the basis vectors in two partitions when the partial flag is extended to a full flag and given a basis.

Define $\{\delta_1, ..., \delta_k\}$ as $\delta_1 = d_1$ and $\delta_i = d_i - d_{i-1}$ for $i > 1$. These can be thought of as the sizes of the partitions.

Definition 3.3. *Let C_k be the set of symmetric matrices $\{c_{i,j}\}_{i,j=1}^{k}$ such that for all i, $\sum\limits_{j=1}^{k} c_{i,j} = \sum\limits_{j=1}^{k} c_{j,i} = \delta_i$ and $2 \mid c_{i,i}$.*

Proposition 3.4. *For every partial flag F_P, $c(F_P) \in C_k$.*

Proof. We first prove $\sum\limits_{j=1}^{k} c_{i,j} = \delta_i$. By Lemma 3.1, letting $j = 0$, we get $d_{i,0} = \sum\limits_{r=1}^{i} \sum\limits_{s=1}^{k} c_{r,s}$. But we know $d_{i,0} = \dim(V_i) = d_i$. Hence $d_i = \sum\limits_{r=1}^{i} \sum\limits_{s=1}^{k} c_{r,s}$ and thus $d_i - d_{i-1} = \sum\limits_{s=1}^{k} c_{i,s}$. A similar argument holds for the other sum. Finally, let F_B be any extension of F_P to G/B, and let $\gamma = c(F_B)$. Since γ is symmetric and its diagonal elements are 0, by Lemma 3.2, it is clear that $c_{i,j} = c_{j,i}$ and $c_{i,i}$ must be even. □

Proposition 3.5. *For every $c \in C_k$, there exists a partial flag F such that $c(F) = c$.*

Proof. Suppose we have a matrix $c \in C_k$. Let the matrix p' be a k^2 by k^2 matrix defined by: for $i, j = \{1, ..., k\}$, $p'_{j+k(i-1),i+k(j-1)} = c_{i,j}$, and all other elements of p' are 0. Thus, p' is a generalized permutation matrix (i.e. it contains exactly one nonzero entry in each row and each column), and if p' is partitioned into k^2 submatrices, each submatrix size k by k, then the (i, j) submatrix contains exactly one nonzero entry, which has value $c_{i,j}$. Now define the exchange matrix E_m as the m by m matrix with 1s along the skew diagonal and 0s elsewhere. Finally, let the matrix p be the block matrix that results when the nonzero entries of p' are replaced by blocks such that an entry with value m is replaced by E_m. Then $p \in C_{2n}$ as described in Proposition 2.2, i.e. it is the permutation matrix of a fixed-point-free involution corresponding to an orbit of K on G/B. Now take any flag F_B in this orbit and project it down to $F_P \in G/P$. By Lemma 3.2, we have that $c(F_P) = c$. An example of these operations is detailed in the subsequent diagram.

$$
\left[\begin{array}{c|c|c} 2 & 1 & 3 \\ \hline 1 & 4 & 1 \\ \hline 3 & 1 & 0 \end{array}\right]
\qquad
\left[\begin{array}{ccc|ccc|ccc}
2 & & & & & & & & \\
 & & & 1 & & & & & \\
 & & & & & & 3 & & \\ \hline
 & 1 & & & & & & & \\
 & & & & 4 & & & & \\
 & & & & & & & 1 & \\ \hline
 & & 3 & & & & & & \\
 & & & & & 1 & & & \\
 & & & & & & & & 0
\end{array}\right]
\qquad
\left[\begin{array}{cccccc|cccccc|cccc}
 & 1 & & & & & & & & & & & & & & \\
1 & & & & & & & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & & & & & & & 1 & \\
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 & & 1 & & & & & & & & & & & & & \\
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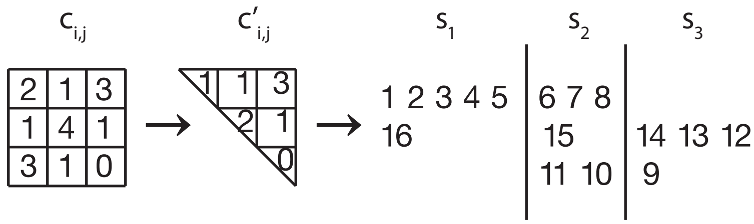
 & & & & & & & 1 & & & & & & & & \\
 & & & & & & & & & & & & & & & 1 \\ \hline
 & & & & & 1 & & & & & & & & & & \\
 & & & & 1 & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & & & & & & & & & & 1 & & & &
\end{array}\right]
$$

$\square$

Definition 3.6. *For each matrix $c \in C_k$, define the corresponding* standard flag $S(c)$ *using the following algorithm. Take only the upper triangular half of c, and halve all the diagonal elements; call the new matrix c'. Let the row sums of c' be $\{a_1, ..., a_k\}$. Sets $s_1, ..., s_k$ with sizes $\delta_1, ..., \delta_k$, such that $\bigcup s_i = \{1, ..., 2n\}$, will be constructed (starting from empty sets) as follows. Let $I = \{1, ..., 2n\}$; elements will be continually removed from I as they are added to various s_i. For i from 1 to k, move the a_i smallest elements of I to s_i (moving n elements in total). Then, for i from 1 to k, for j from i to k, move the $c'_{i,j}$ greatest elements of I to*

s_j (which moves the remaining n elements). Finally, the standard flag $S(c)$ is given by the subspaces $V_{d_i} = \bigoplus_{j=1}^{i} \bigoplus_{s \in s_j} e_s$.

An example of this process is shown below.



The flag $S(c)$ produced is:

$$V_6 = \langle e_1, e_2, e_3, e_4, e_5, e_{16} \rangle$$

$$V_{12} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_{10}, e_{11}, e_{15}, e_{16} \rangle$$

$$V_{16} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16} \rangle$$

Proposition 3.7. *$c(S(c)) = c$ for all c.*

Proof. By inspection of the construction process in Definition 3.6. We want the number of symplectic pairs between partitions i and j to match $c_{i,j}$ so that we can use Lemma 3.2. Consider the ith row of c'. The first elements of a_i symplectic pairs are placed in s_i. The second elements of these pairs are placed in $s_i, ..., s_k$ with frequencies given by $c'_{i,j}$. $\square$

Proposition 3.8. *The projection F_P of any standard full flag F_B to G/P is in the same K-orbit as the standard partial flag $S_P = S(c(F_P))$.*

Proof. Let $\gamma = c(F_B)$, and let $c = c(F_P)$. Define $\{a_1, ..., a_k\}$ and $s_1, ..., s_k$ from c as in Definition 3.6. Let $I = \{1, ..., 2n\}$ to begin. Now consider the basis vectors corresponding to the smallest a_1 elements of I, i.e. $e_1, ..., e_{a_1}$. These vectors go to the subspace V_{d_1} in F_P. Then consider the basis vectors corresponding to the largest a_1 elements of I. These vectors go to various subspaces V_{d_j} in F_P depending on γ. The number of vectors going to the subspace V_{d_j} corresponds exactly to $c'_{1,j}$ due to Lemma 3.2. Then it is possible, through a permutation of $\{2n, ..., 2n - a_1 + 1\}$, to rearrange these vectors so they correspond to their standard locations in F'_P. This permutation can be composed with an identical permutation of $\{1, ..., a_1\}$ (which would not alter F_P) to create a symplectic permutation. Repeating this process with $a_2, ..., a_k$ we obtain a final symplectic permutation matrix that would take F_P to F'_P. □

Proposition 3.9. *For any $F_1, F_2 \in G/P$, $c(F_1) = c(F_2)$ iff F_1 and F_2 are in the same K-orbit.*

Proof. Suppose $c(F_1) = c(F_2)$. Extend F_1 and F_2 to any full flags and use Theorem 2.3 to send these to standard full flags by a symplectic transformation. Then by Proposition 3.8 the projection of these back down to G/P can be sent to the standard partial flag constructed from $\{c_{i,j}\}$ by symplectic transformations. Hence F_1 and F_2 are in the same K-orbit.

Now we show the reverse. Suppose that for some $k \in K$, $kF_1 = F_2$. But K preserves $d(F)$ and thus preserves $c(F)$. Hence, $c(F_1) = c(kF_1)$. □

Theorem 3.10. *The orbits of K on G/P are in bijection with C_k.*

Proof. By Proposition 3.9, the orbits of K on G/P are in bijection with the possible values of $c(F)$ where F ranges over G/P. But by Propositions 3.4 and 3.5, we know that this is precisely C_k. Hence we are done. □

4. Q-orbits on G/P

Keep the definitions of k and δ_i from before, as parameters of the partial flag variety G/P. The following will define $C_k^\bullet$, whose elements parameterize standard partial flags, which we will eventually show to correspond exactly to Q-orbits on G/P.

Definition 4.1. *Define $b_{x\in S}$ as 1 if $x \in S$ and 0 if $x \notin S$. Define $I_k = \{1, ..., k\}$. Then define $C_k^\bullet$ as*

$$\bigsqcup_{s=1}^{k} \bigsqcup_{I_{(X)}\subseteq I_k} \bigsqcup_{I_{(Y)}\subseteq I_k} C_k^-$$

where $|I_{(X)}| = |I_{(Y)}| = s$ and C_k^- is the set of k by k nonnegative symmetric matrices such that for all i, $\sum_{j=1}^{k} c_{i,j}^- = \sum_{j=1}^{k} c_{j,i}^- = \delta_i - b_{i\in I_{(X)}} - b_{i\in I_{(Y)}}$ and $2 \mid c_{i,i}^-$.

Note that the standard flags in this section are different from the standard flags in Section 3. The following will give an algorithm to construct these standard flags.

Definition 4.2. *From an element $(s, I_{(X)}, I_{(Y)}, c_k^-) \in C_k^\bullet$, define a standard (full or partial) flag as follows. Use Definition 3.6 to construct sets $s_1, ..., s_k$ from c_k^-, so that $\bigcup_{i=1}^{k} s_i = \{1, ..., 2(n-s)\}$. Then add s to all the elements in the s_i, so that $\bigcup_{i=1}^{k} s_i = \{s+1, ..., 2n-s\}$. Next, for i from 1 to s, where j is the ith smallest element of $I_{(X)}$, place i in s_j; then for i from 1 to s again, where j is the ith smallest element of $I_{(Y)}$, place $2n+1-i$ in s_j. Finally, define the following basis (similarly to Proposition 2.7):*

$$u_{s+1} = e_{s+1},\ \ ...,\ \ u_{2n-s} = e_{2n-s},$$

$$u_1 = e_1 + e_2,\ \ u_2 = e_1 + e_3,\ \ ...,\ \ u_{s-1} = e_1 + e_s,\ \ u_s = e_1,$$

$$u_{2n} = e_{2n-1},\ \ u_{2n-1} = e_{2n-2},\ \ ...,\ \ u_{2n-s+2} = e_{2n-s+1},$$

$$u_{2n-s+1} = e_{2n} - e_{2n-1} - ... - e_{2n-s+1}$$

Then the flag is given by $V_{d_i} = \bigoplus_{j=1}^{i} \bigoplus_{s\in s_j} u_s$.

The projection of a standard full flag from Proposition 2.7 onto G/P can be written by partitioning the symbols according to the signature of P. For example, let $2n = 4$ and take the standard flag of the G/B orbit $A_2Y_1A_2X_1$, or written with partitions, $A_2|Y_1|A_2|X_1|$. Projecting this to G/P with signature $(2, 4)$, we obtain $A_2Y_1|A_2X_1|$.

Clearly the order of symbols within a partition does not matter, since the flags are the same up to permutation of basis vectors in a partition. Using the current example, $A_2Y_1|A_2X_1$ is the same flag as $Y_1A_2|A_2X_1$.

Theorem 4.3. *Any standard full flag projected to G/P with more than one X symbol in a partition remains in the same Q-orbit when the first X, and the Y it corresponds to, are replaced by As. Similarly, any standard full flag projected to G/P with more than one Y symbol in a partition can be written such that the second Y, and the X it corresponds to, are replaced by As.*

Proof. Take a projection of a standard full flag to G/P where two Y symbols are in one partition. Let the first symbol be Y_a, corresponding to the standard basis vector e_{2n-a}. The second symbol is Y_{a+1} and corresponds to e_{2n-a-1}. Then X_a is e_1+e_{a+1} and X_{a+1} is e_1+e_{a+2}:

$$u_{x_a} = e_1 + e_{a+1}$$

$$u_{x_{a+1}} = e_1 + e_{a+2}$$

$$u_{y_a} = e_{2n-a}$$

$$u_{y_{a+1}} = e_{2n-a-1}$$

Now consider the transformation q:

$$qe_{a+1} = e_{a+1}$$

$$qe_{a+2} = e_{a+1} + e_{a+2}$$

$$qe_{2n-a-1} = e_{2n-a-1}$$

$$qe_{2n-a} = e_{2n-a} - e_{2n-a-1}$$

and $qe_i = e_i$ for the rest. It is clear that $q \in Q_{2n}$.

Then under this transformation, $u_{x'_a} = e_1 + e_{a+1}$ remains the same, $u_{x_{a+1}}$ becomes $u_{x'_{a+1}} = e_1 + e_{a+1} + e_{a+2}$. However, since $u_{x'_{a+1}}$ appears later than $u_{x'_a}$ in the sequence, it is equivalent to say $u_{x'_{a+1}} = e_{a+2}$ as an alternative basis of the same partial flag.

Similarly $u_{y'_a} = e_{2n-a}$ becomes $e_{2n-a} - e_{2n-a-1}$ and $u_{y'_{a+1}} = e_{2n-a-1}$ remains unchanged, but since $u_{y'_a}$ and $u_{y'_{a+1}}$ are in the same partition, it is equivalent to say $u_{y'_a} = e_{2n-a}$ as an alternative basis of the same partial flag.

So now we have

$$\begin{aligned} u_{x'_a} &= e_1 + e_{a+1} \\ u_{x'_{a+1}} &= e_{a+2} \\ u_{y'_a} &= e_{2n-a} \\ u_{y'_{a+1}} &= e_{2n-a-1} \end{aligned}$$

and a symplectic permutation matrix will transform this into a standard basis, with X'_{a+1} and Y'_{a+1} replaced by an A pair. A similar argument holds for two X in the same partition. □

Proposition 4.4. *The projection F_P of any standard full flag F_B to G/P is in the same Q-orbit as some standard partial flag F'_P constructed by Definition 4.2.*

Proof. Remove the XY pairs from the symbolic representation of F_B; this reduces $(s, I_{(X)}, I_{(Y)}, c^-_{2n})$ to $(0, \varnothing, \varnothing, c^-_{2n})$. Then we are left with a standard flag for K-orbits (from Definition 3.6). By application of Theorem 3.8, when we project this to G/P we can multiply by a symplectic permutation matrix to get a standard partial flag for K-orbits. When the XY pairs are reinserted and F_B is projected into G/P, we may use Theorem 4.3 to transform F_P until there is at most one X and one Y symbol in each partition. Then we can construct $I_{(X)}$ and $I_{(Y)}$ with no repeated elements within each set. □

Let the partitions of a partial flag be numbered (indexed) as $(1, 2, ..., k)$.

Define the W-position of a partial flag as the boolean matrix

$$\mathrm{relpos}_W(F) = \{V_{d_i} \cap V_{d_{j-1}}^{\perp} \overset{?}{\subset} W\}_{i,j}.$$

Proposition 4.5. *For a standard partial flag, let $I_{(X)} = \{x_1 < ... < x_s\}$ and $I_{(Y)} = \{y_1 < ... < y_s\}$. Then the W-position of the partial flag is given by: $V_{d_i} \cap V_{d_{j-1}}^{\perp} \subset W$ iff $i \geq x_t$ and $j \leq y_t$ for some t.*

Proof. We induct by starting with G/B and projecting down to G/P in small increments. Construct sets $s_1, ..., s_k$ from Definition 4.2. Then sort each set and concatenate them, creating an ordered $2n$-tuple $(s'_1, ..., s'_{2n})$ (so the smallest element of s_1 is s'_1, the next smallest is s'_2; the smallest element of s_2 is $s'_{|s_1|+1}$, and so on). Construct a full flag F_B by $V_i = \bigoplus_{j=1}^{i} u_{s'_j}$ (the vectors $(u_1, ..., u_{2n})$ also given by 4.2). Thus the projection of F_B to G/P is F_P. The result of the proposition for the full flag F_B is shown in Matsuki (see [5], Equation 3.1).

Suppose the signature of the final parabolic P is $(d_1, ..., d_k)$, and let $(d'_1, ..., d'_{2n-k})$ be the complement in $\{1, ..., 2n\}$ (such that $d'_1 < ... < d'_{2n-k}$). Let P_l be the parabolic with signature $\{1, ..., 2n\} \backslash \{d'_1, ..., d'_l\}$. By inductive hypothesis P_{l-1} has W-position given by the proposition. Let p_l be the index of the partition containing d'_l (as its highest dimension) in G/P_{l-1}. When projecting down from P_{l-1} to P_l, the row and the column indexed p_l in $\mathrm{relpos}_W(F_{P_{l-1}})$ are deleted, and thus $\mathrm{relpos}_W(F_{P_l})$ can be described by the same (x_t, y_t) except that all $x_t > p_l$ and $y_t > p_l$ are decremented by 1. But notice that due to the re-indexing of partitions, all X and Y symbols that appeared in a partition higher than p_l have their partition indexed decremented by 1 when projected. Thus the X and Y symbols' partition indices still correspond to the x_t and y_t when projected down. Finally, note that we always have $x_1 < ... < x_s$ and $y_1 < ... < y_s$, since F_B was constructed so that its projection into G/P had at most one X and one Y in each partition. The induction is complete. $\square$

This leads us to the main result of this section.

Theorem 4.6. *The orbits of Q on G/P are in bijection with $C_k^{\bullet}$ (as defined by 4.1).*

Proof. First we prove injectivity. It suffices to show that every $F \in G/P$ is in the same Q-orbit as some standard partial flag. Extend F to a full flag, use Proposition 2.7 to send it to a standard full flag, project this down to G/P, and use Proposition 4.4 to send the projection to a standard partial flag.

Next we prove surjectivity. It suffices to show that no two standard partial flags are in the same Q-orbit. Standard partial flags are parameterized by $(s, I_{(X)}, I_{(Y)}, c_k^-)$, but Theorem 4.5 shows that different $I_{(X)}$ or $I_{(Y)}$ implies different W-positions, implying different Q-orbits (since Q preserves the property of a subspace being in W). If c_k^- is different then that implies different K-orbits, implying different Q-orbits. $\square$

Note that if $P = B$, we recover Theorem 2.5. In this case $\delta_1 = ... = \delta_{2n} = 1$. Then for some i, if $i \in I_{(X)}$ and $i \in I_{(Y)}$ simultaneously, C'_k becomes the null set; otherwise, for $i \in I_{(X)} \cup I_{(Y)}$, $\sum_{j=1}^{k} c_{i,j} = \sum_{j=1}^{k} c_{j,i} = 0$ and those rows and columns of the matrix may simply be removed, leaving us with an element of $C(I_{(A)})$.

Finally, the orbits of Q on G/P are equivalent to the orbits of K on $V^* \times G/P$, where V^* denotes $\mathbb{C}^{2n}\backslash\{0\}$; and the orbits of K on G/P are equivalent to the orbits of K on $\{0\} \times G/P$. Thus the orbits of K on $V \times G/P$ consist of the orbits of Q on G/P, plus the orbits of K on G/P. Combining Theorem 3.10 and Theorem 4.6, we obtain:

Corollary 4.7. *The orbits of K on $V \times G/P$ are in bijection with*

$$\bigsqcup_{s=0}^{k} \bigsqcup_{I_{(X)} \subseteq I_k} \bigsqcup_{I_{(Y)} \subseteq I_k} C'_k$$

where we retain all definitions from Theorem 4.6.

5. Conclusion

We have computed the orbits of K on G/P and the orbits of K on $V \times G/P$. The bijections for these results take the form of a nonnegative integer matrix for each orbit, reminiscent of the left hand side of the classical RSK correspondence; thus the next step would be to relate these orbits to semistandard Young bi-tableaux. It is not necessarily true that the orbits biject directly to semistandard bi-tableaux, and it remains to be explored what modifications, if any, must be made. Another direction would be to compute the closure order for these orbits, i.e. the orbits that are contained in the closure of other orbits, which admits a partial order akin to the Bruhat order.

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