Lowest-weight representations of Cherednik algebras in positive characteristic

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January 29, 2012

1 Introduction

Lowest-weight representations of Cherednik algebras $H_{\hbar,c}$ have been studied in both characteristic 0 and positive characteristic. However, the case of positive characteristic has been studied less, because of a lack of general tools. In positive characteristic the lowest-weight representation $L_c(\tau)$ of the Cherednik algebra is finite-dimensional. The representation theory of complex reflection groups becomes more complicated in positive characteristic, which makes the representation theory of the associated Cherednik algebras more interesting.

The Cherednik algebra for the rank 1 group $\mathbb{Z}/l$ has been studied by Latour in [Latour]. Martina Balagović and Harrison Chen have studied the Cherednik algebras for $GL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$ in [BC]. The nonmodular case for the symmetric group S_n (where the characteristic does not divide the order of the group) has been studied by Roman Bezrukavnikov, Michael Finkelberg, and Victor Ginzburg in the context of algebraic geometry in [BFG]. The nonmodular case has also been studied in [Gordon] for the symmetric group S_n (permutation matrices).

Unlike previous work, this paper studies Cherednik algebras for complex reflection groups in characteristic 0 reduced modulo p (the characteristic), mostly in the modular case (where p divides the order of the group) which is the most difficult case. The groups studied here are also of higher rank than previous work. We study representations of Cherednik algebras of complex reflection groups $G(m,m,n)$ and $G(m,1,n)$ where the parameter $\hbar$ for the Cherednik algebra is equal to 0, focusing on finding generators for the submodule J_c and then describing the quotient $M_c/J_c = L_c$ that is a representation of the Cherednik algebra.

2 Definitions

We now define the Cherednik algebra which was introduced in [EG], and give some preliminary definitions. We use [EM] as a reference for this section.

Let $\mathfrak{h}$ be a vector space of dimension n over a field K. A **reflection** is a linear operator s on $\mathfrak{h}$ that has finite order and such that $\text{rank}(s - I) = 1$. A finite subgroup G of $GL(\mathfrak{h})$ is a **complex reflection group** if it is generated by reflections. Complex reflection groups are generally studied when $K = \mathbb{C}$, which is why they are called "complex", but the definitions still make sense for other fields. The family of complex reflection groups we work with is indexed as $G(m,r,n)$: $G(m,r,n)$ is the group of n by n permutation matrices with entries that are m^{th} roots of unity such that the product of the entries is an $(m/r)^{th}$ root of unity. For example, $G(1,1,n)$ is the symmetric group S_n, $G(2,1,n)$ is signed permutation matrices, and $G(2,2,n)$ is signed permutation matrices with an even number of -1s. We note that the characteristic of K cannot divide m.

Let S be the set of reflections in our chosen complex reflection group G. We note that G acts on $\mathfrak{h}^*$ as well as $\mathfrak{h}$: For all $f \in \mathfrak{h}^*, x \in \mathfrak{h}, g \in G$, we have $gf(x) = f(g^{-1}x)$.

Each reflection in G has $n-1$ eigenvectors with eigenvalue 1 and 1 eigenvector with a different eigenvalue that is a root of unity. For each reflection s, we can choose eigenvectors $\alpha_s^\vee \in \mathfrak{h}, \alpha_s \in \mathfrak{h}^*$ for the eigenvalue of s that is not 1. We scale them so that $\langle \alpha_s^\vee, \alpha_s \rangle = 2$.

Select a function c from the conjugacy classes of S to K and a number $\hbar$ in K. $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ the tensor algebra of the direct sum of $\mathfrak{h}$ and $\mathfrak{h}^*$ and $K[G]$ is the group algebra of G. The **Cherednik algebra** $H_{\hbar,c}(G, \mathfrak{h})$ is $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes K[G]$, modulo the following relations:

For any $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$, we set $[x, x'] = 0, [y, y'] = 0$ and

$$[y, x] = \hbar\langle y, x\rangle - \sum_{s \in S} c(s)\langle y, \alpha_s\rangle\langle \alpha_s^\vee, x\rangle s.$$

This algebra is $\mathbf{Z}$-graded: $x \in \mathfrak{h}^*$ have degree 1, $y \in \mathfrak{h}$ have degree -1, and $g \in G$ have degree 0.

Lowest weight $\mathbf{Z}$-graded representations of $H_{\hbar,c}(G, \mathfrak{h})$ are constructed from Verma modules. Let τ be an irreducible representation of G. By the PBW property, the Cherednik algebra can be decomposed as $\mathrm{Sym}(\mathfrak{h}^*) \otimes K[G] \otimes \mathrm{Sym}(\mathfrak{h})$. Then let $\mathrm{Sym}(\mathfrak{h})$ act as 0 on τ to construct the Verma module $M_c(G, \mathfrak{h}, \tau) = H_{\hbar,c}(G, \mathfrak{h}) \otimes_{K[G] \ltimes \mathrm{Sym}(\mathfrak{h})} \tau$. There is a unique maximal proper submodule J_c of M_c which can be realized as the kernel of a particular bilinear form $\beta_c \colon M_c(G, \mathfrak{h}, \tau) \otimes M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*) \to K$, where $\bar{c}$ is the function such that $\bar{c}(s) = c(s^{-1})$. As vector spaces, M_c is isomorphic to $\mathrm{Sym}(\mathfrak{h}^*) \otimes \tau$, which follows from the PBW property.

The **Dunkl operator** of an element $y \in \mathfrak{h}$ acting on M_c is defined as follows:

$$D_y = \hbar\partial_y \otimes 1 - \sum_{s \in S} c(s)\frac{\langle y, \alpha_s\rangle}{\alpha_s}(1 - s) \otimes s$$

For all $f \otimes t \in \mathrm{Sym}(\mathfrak{h}^*) \otimes \tau, x \in \mathfrak{h}^*, g \in G, y \in \mathfrak{h}$:

$$x(f \otimes t) = xf \otimes t$$
$$g(f \otimes t) = g(f) \otimes g(t)$$
$$y(f \otimes t) = D_y(f \otimes t)$$

M_c and L_c inherit a grading from $H_{\hbar,c}(G, \mathfrak{h})$. We say that the Hilbert series of L_c is $\sum_{i=0}^{\infty} (\dim(L_c)_i)t^i$.

β_c has the following properties:

For all $f \otimes t \in M_c, h \in M_{\bar{c}}, v \in \tau, w \in \tau^*, x \in \mathfrak{h}^*, y \in \mathfrak{h}$:

$$\beta_c(xf \otimes t, h) = \beta_c(f \otimes t, xh)$$
$$y(f \otimes t) = D_y(f \otimes t)$$
$$\beta_c(f \otimes t, yh) = \beta_c(D_y(f \otimes t), h)$$
$$\beta_c(v, w) = w(v)$$

These relations give recursive properties for β_c that allow us to calculate the bilinear form and thus find J_c. The lowest weight representation of the Cherednik algebra is $L_c(\tau) = M_c/J_c$.

We see by the properties of β_c, and because J_c is the kernel of β_c, that if the Dunkl operators on an element $v \in M_c$ all send v to 0, then $v \in J_c$. We also see that if the Dunkl operators on an element $w \in M_c$ give elements of J_c, then w is also in J_c. Since taking the Dunkl operator on an element brings its degree down, that allows us to calculate J_c recursively.

Throughout, let $\hbar = 0$ and c be generic: we use a function field, adjoining one variable c_i for each conjugacy class i of S. We focus on three types of complex reflection groups as G: $G(m, m, n), G(m, 1, n)$ for $m > 1$, and $G(1, 1, n) = S_n$, the symmetric group. For $G(m, m, n)$ and τ trivial, we describe some generators of J_c; in the case where p divides n, we are able to then prove that L_c is an irreducible representation of the Cherednik algebra, but for general n we take steps towards this proof. For $G(m, m, 2)$ we completely describe J_c and L_c for all m, p, τ. For $G(m, 1, n)$ and τ trivial, we describe some of the generators of J_c, and completely describe both J_c and L_c for the case $m = 1$ (i.e. S_n) and p divides n.

3 Specht modules and Garnir polynomials

Representations of $G(m, r, n)$ are constructed from the representations of the symmetric groups of smaller size known as Specht modules. When the characteristic of K is 0, Specht modules give all irreducible representations, but in positive characteristic some Specht modules are reducible. Specht modules will be important for the rest of the paper. Our reference for this background information is [Peel]. We omit the full construction of Specht modules since only their realization using Garnir polynomials is important to this paper.

We can consider a polynomial ring in n variables over a field K as a S_n-module, where the S_n action is the permutation of the variables. Specht modules can be realized as submodules of $K[x_1, \ldots, x_n]$.

Specht modules are indexed by partitions of n. For a given partition of n, a **Young tableau** is a filling of the partition with the numbers from 1 through n. An example of a Young tableau for the partition $(4, 2, 1)$ follows:

1	2	3	4
5	6		
7			

A **standard Young tableau** is a Young tableau in which the entries in the rows and columns are increasing top to bottom and left to right. For example, the tableau above is standard.

The **Garnir polynomial** for a Young tableau T is defined as follows. Let $a_{i,j}$ be the entry in the i^{th} row and j^{th} column of T. Let m be the number of columns in T with at least 2 entries. Then

$$f_T(x) = \prod_{1 \le d \le m} \prod_{r < s} (x_{a_{r,m}} - x_{a_{s,m}})$$

is the Garnir polynomial for the Young tableau T. For example, the Garnir polynomial for the above tableau is $(x_1 - x_5)(x_1 - x_7)(x_5 - x_7)(x_2 - x_6)$.

The Specht module for a given partition λ is denoted as S_λ. It can be realized as a subspace of $K[x_1, \ldots, x_n]$ spanned by the Garnir polynomials of the Young tableaux for λ, with the Garnir polynomials for the standard Young tableaux as a basis.

4 $G(m, m, n)$

We consider the Cherednik algebras of the groups $G(m, m, n)$, which are permutation matrices with entries that are m^{th} roots of unity such that all of the entries multiply to 1. We begin by describing the reflections and Dunkl operators related to these groups.

Let μ be a primitive m^{th} root of unity in K.

The reflections for $G(m,m,n)$ have only one conjugacy class when $n \geq 3$ or when $n = 2$ and m is odd. The reflections are indexed as $s_{i,j,\ell}$: x_i goes to $\mu^{-\ell}x_j$ and x_j goes to $\mu^{\ell}x_i$, and all other basis elements are sent to themselves by the operator. We can therefore refer to the function $c(s)$ on the conjugacy classes of the reflections by a single parameter c. We can then describe the Dunkl operators for $G(m,m,n)$ on $M_c(\tau)$ (D_{y_i} is written as D_i):

$$D_i = -c \sum_{\substack{j \neq i \\ 0 \leq \ell < m}} \frac{1}{x_i - \mu^{-\ell}x_j}(1 - s_{i,j,\ell}) \otimes s_{i,j,\ell}$$

However, in the case where $n = 2$ and m is even ($m = 2k$), we have two conjugacy classes of reflections: $s_{i,j,\ell}$ where ℓ is even and $s_{i,j,\ell}$ where ℓ is odd. In this case, the function $c(s)$ can be referred to by two parameters c_0 (for even reflections) and c_1 (for odd reflections), with the Dunkl operators appearing as follows:

$$\begin{aligned} D_i = &-c_0 \sum_{\substack{j \neq i \\ 0 \leq \ell < k}} \frac{1}{x_i - \mu^{-2\ell}x_j}(1 - s_{i,j,2\ell}) \otimes s_{i,j,2\ell} \\ &- c_1 \sum_{\substack{j \neq i \\ 0 \leq \ell < k}} \frac{1}{x_i - \mu^{-2\ell-1}x_j}(1 - s_{i,j,2\ell+1}) \otimes s_{i,j,2\ell+1} \end{aligned}$$

Lemma 4.1. *If $n \equiv i \mod p$ for $1 \leq i \leq p$, then the squarefree monomials of degree i are killed by the Dunkl operators for $G(m,m,n)$.*

Proof. Using the fact that $\sum_{k=0}^{m-1} \mu^k = 0$, we claim that the action of the Dunkl operators on the squarefree monomials of degree i gives 0. If the monomial in question is $x_{e_1} \cdots x_{e_i}$, there are two cases: D_k when $k \notin \{e_j\}$ and D_{e_j} for some $1 \leq j \leq i$.

In the first case, the only $s_{k,r,\ell}$ that are relevant are for $r \in \{e_j\}$. The reflection $s_{k,e_j,\ell}$ produces the term $-\mu^l \frac{x_{e_1} \cdots x_{e_i}}{x_{e_j}}$. If the sum of this is taken over all e_j and l, the result must be 0.

In the second case, we take D_{e_1} as a representative. Reflections of the form $s_{e_1,e_j,\ell}$ produce a term of 0 because that element of the group leaves the monomial unchanged. Reflections of the form $s_{e_1,r,\ell}$ produce $x_{e_2} \cdots x_{e_i}$ for $r \notin \{e_j\}$. There are $n - i$ such r. Therefore, the sum of all such terms is $(n-i)(m)x_{e_2} \cdots x_{e_i}$, which comes out to 0 since $n \equiv i \mod p$. □

4.1 $G(m,m,n)$, $n \equiv 0 \pmod p$.

The first case we consider is $G = G(m,m,n)$ in characteristic p where p divides n but not m. We assume τ is the trivial representation.

Proposition 4.2. *The ideal J is generated by the differences of the m^{th} powers of the x_i and the squarefree monomials of degree p.*

Proof. The differences of the m^{th} powers of the x_i are killed by the Dunkl operators. By Lemma 4.1, the squarefree monomials of degree p are killed, since if p divides n, $n \equiv p \mod p$.

In this case, the highest degree existing in A/J is $(p-1)m$. The basis for this top degree is one element: $x_n^{(p-1)m}$. Since $D_n(x_n^s) = -c(n-1)mx_n^{s-1} = cmx_n^{s-1}$, we know that $\beta(x_n^{(p-1)m}, x_n^{(p-1)m}) =$

$(cm)^{(p-1)m}$. To finish the proof, it is enough to know that the socle is concentrated in top degree, which is the content of Lemma 4.3 (recall that the socle is

$$\mathrm{soc}(A/I) = \{x \in A/I \mid xy = 0 \text{ for all } y \in (A/I)_{>0}\}$$

where $A = M_c$ and $I = J_c$). □

Lemma 4.3. *The algebra A/J is Gorenstein.*

Proof. Any monomial in A/J can be expressed as $x_{e_1}^{a_1} \cdots x_{e_{p-1}}^{a_{p-1}}$ where $1 \leq e_1 < \ldots < e_{p-1} \leq n$ and $0 \leq a_i < m$ for all i. This is because the m^{th} powers of the variables can be effectively taken as equal because the differences of the m^{th} powers are in J. (Monomials that cannot be expressed this way can be expressed with p indices and therefore must be in J.) Multiplying this monomial by $x_{e_1}^{m-a_1} \cdots x_{e_{p-1}}^{m-a_{p-1}}$ yields $x_{e_1}^{m} \cdots x_{e_{p-1}}^{m}$ which is equivalent to $x_n^{(p-1)m}$, which is in A/J. The monomial $x_{e_1}^{m-a_1} \cdots x_{e_{p-1}}^{m-a_{p-1}}$ is also in A/J because it fits the necessary restrictions on indices and coefficients. Therefore, the socle must be in top degree and A/J is Gorenstein. □

4.2 $G(m, m, n)$, $n \equiv i \mod p$, $i \neq 0$

The case where p does not divide n for τ trivial is more difficult.

Proposition 4.4. *The ideal J contains the elementary symmetric functions of the m^{th} powers of the x_j and by the squarefree monomials of degree i.*

Proof. By Lemma 4.1, the squarefree monomials of degree i are killed by the Dunkl operators and thus must be in J. The action of the Dunkl operators on the elementary symmetric functions of the m^{th} powers of the x_i must also be 0 because they are invariants of G. □

Remark 4.5. In the case when $i = 1$ the Hilbert series of the quotient A/J is 1 because the squarefree monomials of degree 1 are the x_i. □

If we mod out by all squarefree monomial ideals of degree i, then we get a Cohen–Macaulay algebra of Krull dimension $i - 1$ because its support is the set of points $(x_1, \ldots, x_n)$ such that at least $n - i + 1$ coordinates are equal to 0. This is a union of $\binom{n}{i-1}$ linear spaces, so the degree of this variety is $\binom{n}{i-1}$. In fact, this algebra has a linear free resolution [ER, Theorem 3], and hence is a level algebra – when a level algebra is quotiented by a regular sequence, the socle of the quotient is in the top degree. Write its Hilbert series as

$$\frac{H(t)}{(1-t)^{i-1}}.$$

Then $\deg H = i - 1$ and $H(1) = \binom{n}{i-1}$.

The elementary symmetric functions $e_1(x^m), \ldots, e_{i-1}(x^m)$ (without using mth powers) form a homogeneous system of parameters since the solution set of n-tuples $(x_1, \ldots, x_n)$ such that the squarefree monomials on the n-tuple and the elementary symmetric functions on the n-tuples are all equal to 0 is simply the single n-tuple $(0, \ldots, 0)$. Since the algebra is Cohen–Macaulay, a homogeneous system of parameters is a regular sequence. So the Hilbert series of A/J is

$$H(t) \cdot \prod_{j=1}^{i-1} \frac{1 - t^{jm}}{1 - t}.$$

Proposition 4.6. *In the case $i = 2$, the elementary symmetric functions of the x_k^m and the squarefree monomials of degree i generate J_c, and the resulting quotient $M_c/J_c = L_c$ is an irreducible representation of the Cherednik algebra.*

Proof. For $i = 2$, we have $H(t) = 1 + (n-1)t$. So the Hilbert series of A/J is

$$(1 + (n-1)t)(1 + t + \cdots + t^{m-1}) = 1 + nt + nt^2 + \cdots + nt^{m-1} + (n-1)t^m.$$

A basis for the degree i part ($1 \leq i \leq m-1$) is given by $x_1^i, \ldots, x_n^i$, so the corresponding representation is the reflection representation where each entry of the matrices are raised to the ith power. This representation is irreducible. The representation in degree m is spanned by $x_1^m, \ldots, x_n^m$ modulo $x_1^m + \cdots + x_n^m$, which is also irreducible. The value of $D_1(x_1^s)$ is $-(c)(n-1)(m)x_1^{s-1}$, therefore we know that $\beta(x_1^m, x_1^m) = (-cm)^m$ since $n - 1 \equiv 1$ mod p, therefore β is nonzero on the degree m part of L_c and therefore L_c is irreducible as a representation of the Cherednik algebra. □

For the general $n \equiv i \mod p$ case, set $r_k = \sum_j x_j^{km}$.

Theorem 4.7. *For any field K, regardless of characteristic, the quotient ring of $K[x_1, \ldots, x_n]$ by the squarefree monomials of degree i and $r_1, \ldots, r_{i-1}$ has top degree $\binom{i}{2}m$ with a basis $x_{j_1}^m x_{j_2}^{2m} \cdots x_{j_{i-1}}^{(i-1)m}$ where $1 < j_1 < \ldots < j_{i-1}$.*

Proof. Let Q be the quotient ring of $K[x_1, \ldots, x_n]$ by the squarefree monomials of degree i. Then the basis for degree d of Q is:

$$\begin{aligned}
&x_j^d (1 \leq j \leq n)\\
&x_{j_1}^k x_{j_2}^{d-k} (0 < k < d, 1 \leq j_1 < j_2 \leq n)\\
&\vdots\\
&x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-1}}^{k_{i-1}} \quad (0 < k_1, k_2, \ldots, k_{i-1}, \sum_s k_s = d, 1 \leq j_1 < j_2 < \cdots < j_{i-1} \leq n)
\end{aligned}$$

We want to find the basis for the top degree of the quotient of Q by $r_1, \ldots, r_{i-1}$. The top degree is $\binom{i}{2}m$. We can find a basis by multiplying the r_s by elements of Q in an appropriate degree. This will allow us to generate the ideal formed in Q by the r_s. We can then eliminate the leading term not already eliminated of each of these polynomials (using lexicographic order) from the generating set for that degree, since it can be expressed as the negative of the sum of the other terms of the polynomial. This should result in a basis for the desired degree of Q quotiented by the r_s. We refer to the quotient of Q by the r_s as V. Since the proof is lengthy, we first demonstrate a small example then show the general case.

Example 4.8. For the small example, assume $i = 4, m = 3$. The top degree of V is then 18, and a basis for Q_d is:

$$\begin{aligned}
&x_j^d \quad (1 \leq j \leq n)\\
&x_{j_1}^k x_{j_2}^{d-k} \quad (0 < k < d, 1 \leq j_1 < j_2 \leq n)\\
&x_{j_1}^{k_1} x_{j_2}^{k_2} x_{j_3}^{k_3} \quad (0 < k_1, k_2, k_3, \sum_s k_s = d, 1 \leq j_1 < j_2 < j_3 \leq n)
\end{aligned}$$

We first consider the elements eliminated from this generating set by multiplying the r_s with terms that use exactly 3 variables. We begin with r_1: this takes the form $x_{j_1}^{k_1} x_{j_2}^{k_2} x_{j_3}^{k_3} * r_1$. The first term not already eliminated would be $x_{j_1}^{k_1+3}x_{j_2}^{k_2}x_{j_3}^{k_3}$, since the previous terms would use 4 variables and then already be eliminated by the squarefree monomials. Therefore, we can eliminate all terms using exactly 3 variables where the first exponent is greater than 3 from the generating set. We then continue with r_2: the first term that uses 3 variables or less is $x_{j_1}^{k_1+6}x_{j_2}^{k_2}x_{j_3}^{k_3}$. However, this has already been eliminated above, so we proceed to the next term using 3 or less variables, which is $x_{j_1}^{k_1}x_{j_2}^{k_2+6}x_{j_3}^{k_3}$. We can then eliminate all terms using exactly 3 variables where the second exponent is greater than 6 from the generating set. By similar logic, the first term not already eliminated that r_3 generates would be $x_{j_1}^{k_1}x_{j_2}^{k_2}x_{j_3}^{k_3+9}$, so we then get an upper limit of 9 for the third exponent in terms that use 3 variables. Since the top degree is 18, we must have the exponent equal to the upper limit for each variable. Then the terms using exactly three variables in the generating set take the form $x_{j_1}^3x_{j_2}^6x_{j_3}^9$ for some $1 \leq j_1 < j_2 < j_3 \leq n$.

We then consider the elements eliminated by multiplying the r_s with terms using exactly 2 variables. We begin with $x_{j_1}^{k_1}x_{j_2}^{k_2} * r_2$. The first term of this polynomial is $x_1^6x_{j_1}^{k_1}x_{j_2}^{k_2}$; however, this has already been eliminated, as has $x_2^6x_{j_1}^{k_1}x_{j_2}^{k_2}$ and so forth. Then the first term that has not already been eliminated is $x_{j_1}^{k_1+6}x_{j_2}^{k_2}$. Therefore, we can eliminate from the generating set all terms using exactly two variables where the first exponent is greater than 6. We then consider $x_{j_1}^{k_1}x_{j_2}^{k_2} * r_3$. The first term of this is $x_1^9x_{j_1}^{k_1}x_{j_2}^{k_2}$, which has already been eliminated, similarly to $x_2^9x_{j_1}^{k_1}x_{j_2}^{k_2}$ and so forth. The next term would be $x_{j_1}^{k_1+9}x_{j_2}^{k_2}$, which has just been eliminated above. $x_{j_1}^{k_1}x_{j_1+1}^{9}x_{j_2}^{k_2}$ was eliminated earlier as well, like $x_{j_1}^{k_1}x_{j_1+2}^{9}x_{j_2}^{k_2}$ and so forth. Therefore, the next term to be eliminated would be $x_{j_1}^{k_1}x_{j_2}^{k_2+9}$. We then see that the upper limit for the second exponent of terms using exactly two variables in the generating set would be 9. We then see that the maximum degree for a term using exactly two variables in the generating set is $6 + 9 = 15$, which is too small. Therefore, all terms with two variables in the generating set have been eliminated.

Finally, we consider the elements eliminated by multiplying the r_s with terms using exactly 1 variable, beginning with the elements of the generating set eliminated by $x_j^k * r_1$. Since all terms of two variables have been eliminated already, the first term of this polynomial that has not been eliminated is x_j^{k+3}. Therefore, all terms of one variable with exponent greater than 3 can be eliminated from the generating set. However, all terms of one variable in the generating set must have an exponent of 18, since that is the top degree, so all terms of one variable have been eliminated.

The only terms then remaining are $x_{j_1}^3x_{j_2}^6x_{j_3}^9$ for some $1 \leq j_1 < j_2 < j_3 \leq n$. □

We now consider the general case. The basis for any degree of Q has terms using exactly 1 variable, 2 variables, up to $i-1$ variables. We begin by considering the elements eliminated from the generating set by multiplying the r_s with the terms with $i-1$ variables. We begin with r_1. This would take the form $x_{j_1}^{k_1}x_{j_2}^{k_2}\cdots x_{j_{i-1}}^{k_{i-1}} * r_1$. The first term of this polynomial not already eliminated would be $x_{j_1}^{k_1+m}x_{j_2}^{k_2}\cdots x_{j_{i-1}}^{k_{i-1}}$, since the previous terms involve i variables and thus must already be equal to 0. By multiplying r_1 with all the polynomials with $i-1$ variables in the basis for degree$\binom{i}{2}m - m$ of Q, we thus can eliminate from the basis of the top degree of V all terms with $i-1$ variables with the exponent of the first variable greater than m. We then continue with r_2. Multiplying it with all the polynomials with $i-1$ variables in the basis for degree$\binom{i}{2}m - 2m$ of Q, the first term in each of these polynomials with less than i variables would be $x_{j_1}^{k_1+2m}x_{j_2}^{k_2}\cdots x_{j_{i-1}}^{k_{i-1}}$. However, this has already been eliminated, since it has $i-1$ variables and the exponent of the first is greater than m. Therefore, we take the next term with less than i variables, which is $x_{j_1}^{k_1}x_{j_2}^{k_2+2m}\cdots x_{j_{i-1}}^{k_{i-1}}$. We then eliminate from the generating set of the top degree of V all terms with $i-1$ variables with the exponent of the second variable greater than $2m$. By similar logic, we can eliminate from the generating set of the top degree of V all terms with $i-1$ variables with the exponent of the s^{th} variable greater than sm by using the polynomials generated by r_s.

We now show that all elements for the top degree of V with $i-2$ variables are eliminated. These are eliminated by $r_2, \ldots, r_{i-1}$ multiplied by the basis elements of the corresponding degree of Q with $i-2$ variables. We begin with $x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}} * r_2$ where $\sum k_s = \binom{i}{2} - 2m$, which generates the term $x_1^{2m} x_{j_1}^{k_1+m} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$. However, we can see that this term was eliminated above. The next term that has not already been eliminated is $x_{j_1}^{k_1+2m} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$. Then we can eliminate from the generating set of the top degree of V all terms with $i-2$ variables with the exponent of the first variable greater than $2m$. We then consider the terms generated in this degree by r_3, which come from $x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}} * r_3$ where $\sum k_s = \binom{i}{2} - 3m$. The leading term of this polynomial is $x_1^{3m} x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$. However, this was already eliminated above. The next term that was not eliminated by the terms with $i-1$ variables above is $x_{j_1}^{k_1+3m} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$. However, that was eliminated by r_2 and the terms with $i-2$ variables. Then the next term that has not already been eliminated by r_2 and the terms with $i-2$ variables is $x_{j_1}^{k_1} x_{j_1+1}^{3m} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$. However, this was eliminated by the terms with $i-1$ variables above. The next term that has not been eliminated is then $x_{j_1}^{k_1} x_{j_2}^{k_2+3m} \cdots x_{j_{i-2}}^{k_{i-2}}$. Then we can remove from the generating set of the top degree of V all terms with $i-2$ variables such that the second exponent is greater than $3m$. Similarly, by an inductive process, we can remove from the generating set of the top degree of V all terms with $i-2$ variables such that the s^{th} exponent is greater that $sm+m$. We then see that all terms with $i-2$ variables in the basis of the top degree of V must be of the form $x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-2}}^{k_{i-2}}$ where $\sum k_s \leq 2m + 3m + \cdots + (i-1)m = \binom{i}{2}m - m$. However, we must have $\sum k_s = \binom{i}{2}m$ since this term is in the top degree of V. Therefore all terms with $i-2$ variables are eliminated from the top degree of V.

We now show by induction that all terms with less than $i-1$ variables are eliminated from the generating set of the top degree of V. The base case is $i-2$ variable terms, which has already been shown. We now show that if all terms with $e+1$ variables have been eliminated, all terms with e variables have been as well.

The terms with e variables in the generating set of the top degree of V are eliminated by each r_s multiplied by the terms of the basis of the corresponding $(\binom{i}{2} - sm)$ degree of Q with e variables. We see that $r_1 * x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_e}^{k_e}$ has $x_{j_1}^{k_1+m} x_{j_2}^{k_2} \cdots x_{j_e}^{k_e}$ as its first term with less than $e+1$ variables - meaning that it is also the first term that has not already been eliminated. Therefore, we can eliminate this term. Then we can remove from the generating set of the top degree of V all of the terms with e variables and the first exponent greater than m. We then consider $r_2 * x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_e}^{k_e}$, which has as its first term that has not already been eliminated $x_{j_1}^{k_1} x_{j_2}^{k_2+2m} \cdots x_{j_e}^{k_e}$. Then we can eliminate all of the terms with e variables and the second exponent greater than $2m$. Similarly, through induction, we can eliminate all terms with e variables and the s^{th} exponent greater than sm. Then the terms remaining are $x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_e}^{k_e}$ where $\sum k_s = m + \cdots + em = \binom{e}{2}m$. However, we must have $\sum k_s = \binom{i}{2}m$, so all terms with e variables are not in the basis.

Then the only terms left in the generating set are $x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-1}}^{k_{i-1}}$ where $k_s \leq sm$ for all s. However, we must have $\sum k_s = \binom{i}{2}m$, so we must actually have $k_s = sm$ for all s. Then our generating set becomes $x_{j_1}^{m} x_{j_2}^{2m} \cdots x_{j_{i-1}}^{(i-1)m}$.

We then consider the terms eliminated by multiplying r_1 by the terms with $i-2$ variables. These appear as $x_1^m x_{j_1}^{k_1} x_{j_2}^{k_2} \cdots x_{j_{i-1}}^{k_{i-2}}$. We then can eliminate the terms of our generating set with first variable x_1. Our basis is then $x_{j_1}^{m} x_{j_2}^{2m} \cdots x_{j_{i-1}}^{(i-1)m}$ where $1 < j_1 < \ldots < j_{i-1} \leq n$ as desired. $\square$

Any representation of S_n can also be considered a representation of $G(m, m, n)$. This is because we have a surjective map $G(m, m, n) \to S_n$, where the matrices of $G(m, m, n)$ are considered as permutation matrices only, with all roots of unity replaced by 1. This allows the elements of $G(m, m, n)$ to act on representations of S_n using this mapping.

The reflection representation for S_n is the standard n-dimensional representation, with the S_n–action as the permutation of the variables, quotiented by the sum of the variables. It is therefore $n-1$ dimensional. The exterior powers of this representation are also representations of S_n and thus of $G(m, m, n)$ as well.

Theorem 4.9. *For any field K, regardless of characteristic, the top degree of the quotient ring of $K[x_1, \ldots, x_n]$ by the squarefree monomials of degree i and the power sums $r_1, \ldots, r_{i-1}$ is isomorphic to the Specht module $S_{(n-i+1,1^{i-1})}$, which is the exterior power $\bigwedge^{i-1} \mathfrak{h}$ where $\mathfrak{h}$ is the $n-1$ dimensional reflection representation of the symmetric group.*

Proof. The quotient ring is defined when $K = \mathbb{Z}$ and then reduced modulo p. To check that it is an exterior power, we can work over $\mathbb{Q}$, so we assume $K = \mathbb{Q}$. Without loss of generality, we can assume $m = 1$. For any partition λ, the function (described in [S]) $b(\lambda) = \sum(j-1)\lambda_j$ denotes the lowest degree of $\mathbb{Q}[x_1, \ldots, x_n]$ in which S_λ appears. We see that $b((n-i+1, 1^{i-1})) = \binom{i}{2}$. (The notation $(n-i+1, 1^{i-1})$ refers to the partition (n-i+1,1,1,...,1) where there are $i-1$ 1s.) Let Q be the quotient of $K[x_1, \ldots, x_n]$ by the squarefree monomials of degree i. We see that the $S_{(n-i+1,1^{i-1})}$ in degree $\binom{i}{2}$ of $K[x_1, \ldots, x_n]$ has as its basis the Garnir polynomials $(x_{e_1} - x_{e_2})(x_{e_1} - x_{e_3})(x_{e_1} - x_{e_4}) \cdots (x_{e_{i-1}} - x_{e_i})$ where $(e_1, \ldots, e_i)$ is the first column of a standard filling for $(n-i+1, 1^{i-1})$. When multiplying this out, there are terms with less than i indices, so it is not killed by the squarefree degree i monomials. Therefore, $chP_{\binom{i}{2}} = \chi_{(n-i+1,1^{i-1})} + \cdots$ for some unknown additional addends. Let V be the quotient of Q by the r_j. We then see that $chV_{\binom{i}{2}} = \sum_{\alpha=1}^{i-1} ch\left(\langle r_\alpha \rangle \otimes V_{\binom{i}{2}-\alpha}\right) + \cdots$. The image of r_α, or $\langle r_\alpha \rangle$, is equivalent to the trivial representation, while $V_{\binom{i}{2}-\alpha}$ is a sum of characters of irreducibles. However, since $b((n-i+1, 1^{i-1})) > \binom{i}{2} - \alpha$, none of these irreducibles can be $S_{(n-i+1,1^{i-1})}$. Therefore, we must have $chV_{\binom{i}{2}} = \sum_{\alpha=1}^{i-1} ch\left(\langle r_\alpha \rangle \otimes V_{\binom{i}{2}-\alpha}\right) + \chi_{(n-i+1,1^{i-1})} + \cdots$. Since the images of the r_j are killed by quotienting by them, we then have that $V_{\binom{i}{2}}$ contains the Specht module $S_{(n-i+1,1^{i-1})}$. Because we know the basis for $V_{\binom{i}{2}}$ by Theorem 4.7 we know the dimensions of both vector spaces are equal to $\binom{n-1}{i-1}$. Therefore, they are equal. □

Remark 4.10. When p does not divide n, $\bigwedge^{i-1} \mathfrak{h}$ is an irreducible representation of the symmetric group. Therefore, V is an irreducible representation of $G(m, m, n)$, since the basis is invariant under roots of unity. To show that L_c is an irreducible representation of the Cherednik algebra, it remains to show that β_c is nonzero on the top degree part, which is V. We have not yet been able to show this. □

4.3 $G(m, m, 2)$ with non-trivial τ (non-modular)

Unlike the general case described previously, we can find the generators for J_c and show that L_c is an irreducible representation of the Cherednik algebra for all m, p, τ for the groups $G(m, m, 2)$, otherwise known as the dihedral groups.

The representations of $G(m, m, 2)$ we describe here are indexed as ρ_i for $-1 \leq i < m/2$, as well as ρ_{-2} and ρ_{-3} when m is even.

ρ_i for $-1 \leq i < m/2$ is the same as the standard 2-dimensional representation of the dihedral group, but m^{th} roots of unity act by their i^{th} power. ρ_0 is the trivial representation, ρ_{-1} for m odd is the sign representation, ρ_{-3} for m even is the sign representation, and ρ_{-1} and ρ_{-2} for m even are 1-dimensional variants on the sign representation, described as follows:

For ρ_{-1}, the element of the dihedral group that acts as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the standard representation acts as (1) on ρ_{-1}, and the element of the dihedral group that acts as $\left(\begin{smallmatrix} 0 & \mu^{-1} \\ \mu & 0 \end{smallmatrix}\right)$ on the standard representation acts as (-1) on ρ_{-1}. For ρ_{-2}, the signs of these operations are reversed.

$G(m,m,2)$ has two conjugacy classes when m is even and only one when m is odd. Therefore, in the cases where m is odd, we continue to use c_0 and c_1 with the implicit understanding that they are equal, and the results still apply.

Proposition 4.11. *For $\tau = \rho_i$ for $-3 \leq i \leq 0$, J_c is generated by xy and $x^m + y^m$ and the Hilbert polynomial of L_c is $(t+1)\frac{t^m-1}{t-1}$.*

Proof. The four cases ρ_i where $-3 \leq i \leq 0$ have the same behavior, since they are all one-dimensional. ρ_0 corresponds to the trivial representation, but the other cases produce the same ideal. The ideal is generated by xy and $x^m + y^m$. Since both of these are invariants of the dihedral group and τ is one-dimensional, their Dunkl operators come out to 0. This is a complete intersection, so the Hilbert polynomial of L_c is $(t+1)\frac{t^m-1}{t-1}$. The top degree of the quotient ring is m: it has dimension 1. To show that this is irreducible, we must show that $\beta(x^m, x^m)$ is nonzero.

In the case of ρ_0, we see that $D_x(x^s) = -\frac{m}{2}(c_0 + c_1)x^{s-1}$ for all $s \leq m$. (When m is odd, it is $-mcx^{s-1}$ since $c_0 = c_1$.) Therefore, $\beta(x^m, x^m)$ is equal to $(-\frac{m}{2}(c_0 + c_1))^m$ when m is even or $(-mc)^m$ when m is odd.

In the case of ρ_{-1} for m even only, $D_x(x^s) = -\frac{m}{2}(c_0 - c_1)x^{s-1}$. Therefore, $\beta(x^m, x^m)$ is equal to $(-\frac{m}{2}(c_0 - c_1))^m$.

In the case of ρ_{-2} for m even only, $D_x(x^s) = \frac{m}{2}(c_0 - c_1)x^{s-1}$. Therefore, $\beta(x^m, x^m)$ is equal to $(\frac{m}{2}(c_0 - c_1))^m$.

In the case of ρ_{-1} for m odd, which is the same as ρ_{-3} for m even, $D_x(x^s) = \frac{m}{2}(c_0 + c_1)x^{s-1}$ (when m is odd, this is mcx^{s-1} since $c_0 = c_1$). Therefore, $\beta(x^m, x^m)$ is equal to $(\frac{m}{2}(c_0 + c_1))^m$ when m is even or $(mc)^m$ when m is odd. □

Proposition 4.12. *For the case $\tau = \rho_1$, J_c is generated by $x \otimes e_1, y \otimes e_2, x^3 \otimes e_2, y^3 \otimes e_1$ and the Hilbert polynomial of L_c is $2t^2 + 2t + 2$.*

Proof. Seeing that the Dunkl operators on $x \otimes e_1, y \otimes e_2$ are 0 is trivial. We see that $D_x(x^3 \otimes e_2) = -\frac{m}{2}(c_0 + c_1)xy \otimes e_1$, which is a multiple of $x \otimes e_1$. Similarly, $D_y(x^3 \otimes e_2) = \frac{m}{2}x^2 \otimes e_1$, which is a multiple of $x \otimes e_1$. $D_x(y^3 \otimes e_1) = \frac{m}{2}(c_0 + c_1)y^2 \otimes e_2$ and $D_y(y^3 \otimes e_1) = -\frac{m}{2}(c_0 + c_1)xy \otimes e_2$, which are both multiples of $y \otimes e_2$. We then see that with these four generators for the ideal, the Hilbert polynomial for L_c is $2t^2 + 2t + 2$. The top degree has dimension two, and is spanned by $x^2 \otimes e_2$ and $y^2 \otimes e_1$. We see that this is isomorphic to ρ_1 itself as a representation of the dihedral group, sending $x^2 \otimes e_2$ to e_1 and $y^2 \otimes e_1$ to e_2. Therefore, the top degree is irreducible as a representation of the dihedral group. We see that $\beta(x^2 \otimes e_2, x^2 \otimes e_2) = -(\frac{m}{2})^2(c_0 + c_1)^2$, which is nonzero, so L_c is irreducible as a representation of the Cherednik algebra. □

Proposition 4.13. *For the case $\tau = \rho_i$ where $i > 1$, J_c is generated by $x \otimes e_1, x \otimes e_2, y \otimes e_1, y \otimes e_2$ and the Hilbert polynomial of L_c is 2.*

Proof. The generators are all easily shown to be killed by Dunkl operators. It follows that the Hilbert polynomial of L_c is simply 2. Then L_c is isomorphic to ρ_i, so it is an irreducible representation of the dihedral group, and $\beta(1 \otimes e_1, 1 \otimes e_1) = 1$, so it is an irreducible representation of the Cherednik algebra as well. □

5 $G(m, 1, n)$

We now focus on the case $G = G(m, 1, n)$, which are groups of permutation matrices with entries that are m^{th} roots of unity. We begin by describing the reflections and Dunkl operators in this case.

Let μ be a primitive m^{th} root of unity in K.

The reflections for $G(m, 1, n)$ have m conjugacy classes. The first class is indexed as $s_{i,j,\ell}$: x_i goes to $\mu^{-\ell}x_j$ and x_j goes to $\mu^{\ell}x_i$, and all other basis elements are sent to themselves by the operator. The other classes is indexed as $t_{k,\ell}$ for $0 < \ell < m$ where x_k is sent to $\mu^{\ell}x_k$ and all other basis elements are sent to themselves. We can therefore refer to the function $c(s)$ on the conjugacy classes of the reflections by m parameters c_i, where $c(s_{i,j,\ell}) = c_1$ for all i, j, ℓ and $c(t_{k,\ell}) = c_{\ell+1}$ for all k, ℓ. We can then describe the Dunkl operators for $G(m, 1, n)$ (D_{y_i} is written as D_i).

$$\begin{aligned} D_i = &- \sum_{\substack{j \neq i, \\ 0 \leq \ell < m}} c_1 \frac{1}{x_i - \mu^{-\ell}x_j}(1 - s_{i,j,\ell}) \otimes s_{i,j,\ell} \\ &- \sum_{0<\ell<m} c_{\ell+1} \frac{1}{\mu^{\ell}x_i}(1 - t_{i,\ell}) \otimes t_{i,\ell} \end{aligned}$$

5.1 Subspace arrangements.

In this case, some of the generators of J_c can be described by a subspace arrangement.

Let $X_i^{(m)}$ be the subspace arrangement consisting of n-tuples $(x_1, \dots, x_n)$ such that the mth powers of some subset of $n - i$ of the coordinates are equal. Its ideal $I_i^{(m)}$ is given by substituting mth powers for the variables into the Specht module S_λ where λ has $n - i - 1$ columns and all rows have $n - i - 1$ cells except possibly the last row, by [LL] (which deals with the case $m = 1$).

Proposition 5.1. *If $n \equiv i \mod p$ where $0 \leq i \leq p - 1$, then the Dunkl operators for $G(m, 1, n)$ kill the generators of $I_i^{(m)}$.*

Proof. If $i \neq p - 1$ or $i = p - 1$ and $n > 2p - 1$, then $\lambda = (n - i - 1, i + 1)$. Otherwise, we have $i = p - 1$ and $n = 2p - 1$, in which case $\lambda = (p - 1, p - 1, 1)$.

We wish to show that for a standard Young tableau e of λ that the Garnir polynomial of the m^{th} powers of the variables, $f_e(x^m)$ (described in Section 3), is killed by the Dunkl operators. For example, if the first $i + 1$ columns of a filling j of the Young diagram for λ are $\{\{j_1, j_2\}, \{j_3, j_4\}, \dots, \{j_{2i+1}, j_{2i+1}\}\}$, then the desired polynomial is:

$$f_j(x^m) = (x_{j_1}^m - x_{j_2}^m)(x_{j_3}^m - x_{j_4}^m) \cdots (x_{j_{2i+1}}^m - x_{j_{2i+2}}^m).$$

A basis of $I_i^{(m)}$ is provided by $f_e(x^m)$ for all e that are standard Young tableaux, since this is the basis for the Specht module S_λ.

Let the first $i+1$ columns of e be $\{\{e_1, e_2\}, \{e_3, e_4\}, \dots, \{e_{2i+1}, e_{2i+1}\}\}$. When considering the action of the Dunkl operators on $f_e(x^m)$, we consider the two separate cases: D_k where $k \notin \{e_i\}$ and the opposite, when k is in $\{e_i\}$.

In the first case, reflections permuting k with another element not in $\{e_i\}$ will go to 0 directly because of the $(1 - s_{k,i,\ell})$ term. For all $0 \leq j \leq i$, the terms for the reflections $s_{k,e_{2j+1},\ell}$ and $s_{k,e_{2j+2},\ell}$ will cancel each other. $f_e(x^m)$ is invariant under the $t_{k,i}$ since all the variables have been raised to the m^{th} power, so those terms will produce 0 as well. Therefore, $D_k(f_e(x^m)) = 0$.

The second case can be calculated by using D_{e_1} as a representative. $f_e(x^m)$ is invariant under the $t_{k,i}$, so they can be disregarded once again. the sum of the terms generated by the reflections $s_{e_1,e_2,\ell}$ for $0 \leq \ell \leq m-1$ gives the term $2mx_{e_1}^{m-1}(x_{e_3}^m - x_{e_4}^m)\cdots(x_{e_{2i+1}}^m - x_{e_{2i+2}}^m)$. For $1 \leq j \leq i$, the sum of the terms generated by the reflections $s_{e_1,e_{2j+2},\ell}$ and $s_{e_{2j+1},e_1,\ell}$ for $0 \leq \ell \leq m-1$ comes to $mx_{e_1}^{m-1}(x_{e_3}^m - x_{e_4}^m)\cdots(x_{e_{2i+1}}^m - x_{e_{2i+2}}^m)$. For $w \notin \{e_i\}$ - there are $n-2i-2$ such values for w - the sum of the terms from the reflections $s_{e_1,e_w,\ell}$ for $0 \leq \ell \leq m-1$ comes out to $mx_{e_1}^{m-1}(x_{e_3}^m - x_{e_4}^m)\cdots(x_{e_{2i+1}}^m - x_{e_{2i+2}}^m)$. $f_e(x^m)$ is invariant under the $t_{k,i}$ since all the variables have been raised to the m^{th} power, so those terms will produce 0 and we can disregard the parameters $c_2, \ldots, c_m$. We then see that:

$$D_k(f_e(x^m)) = -c_1(n-2i-2+i+2)(mx_{e_1}^{m-1})(x_{e_3}^m - x_{e_4}^m)\cdots(x_{e_{2i+1}}^m - x_{e_{2i+2}}^m),$$

which is 0 since $n = i \pmod p$. When the sum of this is taken over all l, the result is still 0. Therefore, $D_k(f_e(x^m)) = 0$ for all $1 \leq k \leq n$ and e that are standard Young tableaux.

In the case when $n = 2p-1$, $\lambda = (p-1, p-1, 1)$. Then for any standard Young tableau j of λ:

$$f_j(x^m) = (x_{j_1}^m - x_{j_2}^m)(x_{j_1}^m - x_{j_3}^m)(x_{j_2}^m - x_{j_3}^m)(x_{j_4}^m - x_{j_5}^m)\cdots(x_{j_{2p-2}}^m - x_{j_{2p-1}}^m).$$

Again, a basis for the ideal (and S_λ) is provided by all $f_e(x^m)$ where e is a standard Young tableau of λ. Therefore we again need only show that this basis is killed by the Dunkl operators.

Let $\{\{e_1, e_2, e_3\}, \{e_4, e_5\}, \ldots, \{e_{2p-2}, e_{2p-1}\}\}$ be the entries in the columns of e. When considering the action of the Dunkl operators on $f_e(x^m)$ there are two cases: D_{e_k} where $k \in \{1,2,3\}$ and where $k \notin \{1,2,3\}$.

The first case can be calculated using D_{e_1} as a representative. We can again disregard the $t_{k,i}$ since $f_e(x^m)$ is invariant under their action. Therefore we can also disregard the parameters $c_2, \ldots, c_m$. For $0 \leq \ell \leq m-1$, the sum of the terms produced by the reflections $s_{e_1,e_2,\ell}$ is $2mx_{e_1}^{m-1}(x_{e_1}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_4}^m - x_{e_5}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m)$. The reflections $s_{e_1,e_3,\ell}$ give terms that sum to $2mx_{e_1}^{m-1}(x_{e_1}^m - x_{e_2}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_4}^m - x_{e_5}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m)$. The sum of the terms produced by $s_{e_1,e_{2r},\ell}$ and $s_{e_1,e_{2r+1},\ell}$ for $2 \leq r \leq p-1$ and $0 \leq \ell \leq m-1$ is

$$mx_{e_1}^{m-1}(2x_{e_1}^m - x_{e_2}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_4}^m - x_{e_5}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m).$$

We can then sum all of the terms produced and multiply by the necessary coefficient $(-c_1)$ to finish calculating the Dunkl operator:

$$D_{e_1}(f_e(x^m)) = -c_1 mx_{e_1}^{m-1}(p)(2x_{e_1}^m - x_{e_2}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_4}^m - x_{e_5}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m) = 0.$$

The representative for the second case is D_{e_4}. For $0 \leq \ell \leq m-1$, the sum of the terms produced by the reflections $s_{e_1,e_4,\ell}, s_{e_2,e_4,\ell}, s_{e_3,e_4,\ell}, s_{e_5,e_4,\ell}$ is

$$3mx_{e_1}^{m-1}(x_{e_1}^m - x_{e_2}^m)(x_{e_1}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_6}^m - x_{e_7}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m).$$

The sum of the terms produced by the reflections $s_{e_{2r},e_4,\ell}, s_{e_{2r+1},e_4,\ell}$ for $3 \leq r \leq p-1$ and $0 \leq l \leq m-1$ is

$$mx_{e_1}^{m-1}(x_{e_1}^m - x_{e_2}^m)(x_{e_1}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_6}^m - x_{e_7}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m).$$

Therefore the final Dunkl operator is:

$$D_{e_4}(f_e(x^m)) = -c_1 x_{e_1}^{m-1}(p)(x_{e_1}^m - x_{e_2}^m)(x_{e_1}^m - x_{e_3}^m)(x_{e_2}^m - x_{e_3}^m)(x_{e_6}^m - x_{e_7}^m)\cdots(x_{e_{2p-2}}^m - x_{e_{2p-1}}^m) = 0.$$

This shows that for all $1 \leq i \leq n$, $D_i(f_e(x^m)) = 0$ for all e that are standard Young tableaux. □

We then see that the generators of $I_i^{(m)}$ are a subset of the generators of J_c. We conjecture that the remaining generators form a regular sequence on $I_i^{(m)}$. In the following section, we discuss the specific case where $m = 1$, or where $G = S_n$.

6 S_n

In general, Cherednik algebras of S_n behave the same way as those of $G(m,1,n)$, simply assuming $m = 1$. We then have the generators of $I_i^{(1)}$ and a regular sequence on it as the generators for J_c. In the case of $n \equiv 0 \mod p$, we see that the desired λ for $I_i^{(1)}$ is $(n-1,1)$ and that the regular sequence on $I_i^{(1)}$ is x_n^p.

Given $i < j$, let $f_{i,j}$ be equal to $x_i - x_j$.

Proposition 6.1. *The J is generated by $f_{1,n}, f_{2,n}, \dots, f_{n-1,n}$ and x_n^p, and the Hilbert series is*

$$h_{A/J}(t) = 1 + t + \cdots + t^{p-1}.$$

Proof. It is clear that $D_k f_{i,j} = 0$ for all i, j, k, because they generate the Specht module for the partition $(n-1,1)$ as Garnir polynomials. To show that x_n^p is in J, we note that $(x_i - x_j)^p = x_i^p - x_j^p$. For all k, $D_k x_n^p$ is the sum of $\frac{x_i^p - x_j^p}{x_i - x_j}$ over various i and j, so belongs to J.

Let I be the ideal generated by $f_{1,n}, \dots, f_{n-1,n}, x_n^p$. Then A/I is a finite-dimensional vector space, so I is a complete intersection. In particular,

$$h_{A/I}(t) = \frac{1 - t^p}{1 - t} = 1 + t + \cdots + t^{p-1}.$$

We have an isomorphism $A/I \cong K[z]/z^p$ via the map $x_i \mapsto z$. On this representation, x_i from the Cherednik algebra acts by multiplication by z, and elements of S_n act trivially. The commutators $[y_i, x_j] = s_{ij} = 1$ and $[y_i, x_i] = -\sum_{j \neq i} s_{ij} = -(n-1) = 1$ are true by the definitions of the Cherednik algebra. Furthermore, $y_i 1 = 0$. Hence the action of the Cherednik algebra on A/I factors through the Weyl algebra $K[z, \partial_z]$, and A/I is irreducible as a representation of the Weyl algebra. We have already seen that $I \subseteq J$, so we are done. □

7 Future Research

We plan to continue working with Cherednik algebras of complex reflection groups $G(m,r,n)$. We would like to further investigate non-trivial τ for $G(m,m,n)$, and hope to fully catalogue $G(m,m,3)$ as we have for the dihedral groups. In the case of $G(m,m,3)$, an interesting structure similar to a regular sequence, except consisting of matrices, appears in the generators of J_c. Such a "matrix regular sequence" is similar to a traditional regular sequence, only involving the determinants of the matrices. We also want to study $G(m,r,n)$ where $r < m$ and consider exceptional complex reflection groups as well.

8 Acknowledgements

I would like to thank the PRIMES program at MIT for giving me the opportunity to do this work, Professor Pavel Etingof of MIT for suggesting the problem, Steven Sam for supervising my work, and Carl Lian for his help.

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