On zeroth Poisson homology in positive characteristic

Yongyi Chen and Michael Zhang

Abstract

A Poisson algebra is a commutative algebra with a Lie bracket $\{,\}$ satisfying the Leibniz rule. Such algebras appear in classical mechanics. Namely, functions on the phase space form a Poisson algebra, and Hamilton's equation of motion is $\frac{df}{dt} = \{f, H\}$, where H is the Hamiltonian (energy) function. Moreover, the transition from classical to quantum mechanics can be understood in terms of deformation quantization of Poisson algebras, so that Schrödinger's equation $-i\hbar\frac{df}{dt} = [f, H]$ is a deformation of Hamilton's equation.

An important invariant of a Poisson algebra A is its zeroth Poisson homology $HP_0(A) = A/\{A, A\}$. It characterizes densities on the phase space invariant under all Hamiltonian flows. Also, the dimension of $HP_0(A)$ gives an upper bound for the number of irreducible representations of any quantization of A.

We study $HP_0(A)$ when A is the algebra of functions on an isolated quasihomogeneous surface singularity. Over $\mathbb{C}$, it's known that $HP_0(A)$ is the Jacobi ring of the singularity whose dimension is the Milnor number. We generalize this to characteristic p. In this case, $HP_0(A)$ is a finite (although not finite dimensional) module over A^p. We give its conjectural Hilbert series for Kleinian singularities and for cones of smooth projective curves, and prove the conjecture in several cases.

1 Introduction

1.1 Classical mechanics and Poisson algebras.

Poisson algebras appear from the mathematical formalism of classical mechanics. Specifically, in classical mechanics the state of a system is characterized by its position $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and its momentum $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$ (for instance, for a single particle in our usual Euclidean space, $n = 3$). The motion is determined by the energy, or Hamiltonian function $H(\mathbf{x}, \mathbf{p})$ on $\mathbb{R}^{2n}$; for example, for a particle of mass m moving in a centrally symmetric gravitational field (e.g. the Earth revolving around the Sun), one has

$$H(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{p_1^2 + p_2^2 + p_3^2}{2m} - \frac{K}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

where K is a positive constant. Namely, the motion is governed by Hamilton's differential equations $\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}$. So if $f(\mathbf{x}, \mathbf{p})$ is a (differentiable) function on the space of states (called an observable) then by the chain rule, its change in time is determined by the equation $\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \sum \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial x_i}\right)$. The last expression is the operation on functions called the Poisson bracket: $\{f, H\} := \frac{\partial f}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial x_i}$. Thus, Hamilton's equations can be succinctly written as $\frac{df}{dt} = \{f, H\}$. For instance, f is an integral of motion (i.e., does not change in time during motion) if and only if $\{f, H\} = 0$ (e.g., H is an integral of motion, which is nothing but the law of conservation of energy).

It is easy to check that the Poisson bracket is bilinear and has the following properties:

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry);
2. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ (the Jacobi identity);
3. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (the Leibniz rule).

The first two properties mean that the algebra A of functions $f(\mathbf{x}, \mathbf{p})$ on the space of states (i.e., of classical observables) is a Lie algebra with operation $\{-, -\}$, and the third condition says that for each $f \in A$, the operation $\{f, -\}$ on A is a derivation. This observation leads to an abstract definition of a **Poisson algebra**, which is a commutative algebra A with a bilinear operation $\{-, -\}$ called the Poisson bracket, which satisfies properties 1-3. Thus, roughly speaking, a Poisson algebra is an algebra in which one can do classical mechanics: one can write down Hamilton's equations, which would have similar properties to the usual ones discussed above.

There are many interesting examples of Poisson algebras. For instance, if M is a smooth manifold or algebraic variety with a symplectic structure (i.e., a nondegenerate closed differential 2-form) then the algebra $\mathcal{O}(M)$ of functions on M is naturally a Poisson algebra. There are also many singular varieties which are Poisson (i.e., their algebra of functions is a Poisson algebra). For instance, in the above setting, if G is a finite group acting on M preserving the symplectic structure, then $\mathcal{O}(M)^G = \mathcal{O}(M/G)$ is a Poisson algebra. Also, if $Q(x, y, z)$ is a (square-free) polynomial in three variables over a field $\mathbb{F}$, then the formula $\{g, h\} = \det \frac{\partial(f,g,h)}{\partial(x,y,z)}$ (where the right hand side is the Jacobian determinant) defines a Poisson structure on the algebra of polynomials $\mathbb{F}x, y, z]$, which descends to the algebra $\mathbb{F}x, y, z]/(f)$ of functions on the algebraic surface M_f defined by the equation $Q(x, y, z) = 0$.

1.2 Poisson algebras and quantization.

Poisson algebras are also important in the context of quantization, as they help capture the interplay between classical and quantum mechanics. Namely, the algebra of classical observables is commutative; that is, the position function x and momentum function p of a particle satisfy the relation $px = xp$. The advent of quantum mechanics, on the other hand, requires an adjustment to this property. Namely, in quantum mechanics instead of x, p we have the position operator X and momentum operator P satisfying the canonical commutation relation $[X, P] = XP - PX = i\hbar$, where $\hbar$ represents Planck's constant, which in real life is a measurable quantity but mathematically may represent a formal variable. These operators represent the act of observation of position and momentum rather than functions.

Consequently, monomials are interpreted chronologically; e.g. XP denotes measuring momentum and then immediately measuring position. Since X and P do not commute, one cannot know both the momentum and the position of a moving quantum particle; this is just a mathematical restatement of Heisenberg's uncertainty principle. Thus, in quantum mechanics, we have a noncommutative algebra of observables $\hat{A}$, spanned by the monomials in X and P, and the quantum dynamics is governed by the (operator) Schrödinger equation $-i\hbar\frac{df}{dt} = [f, H]$, where H is the quantum Hamiltonian operator.

In this picture, classical mechanics is recovered from quantum mechanics by considering the limit $\hbar \to 0$. Algebraically, this means that our noncommutative quantum algebra $\hat{A}$ degenerates into a Poisson algebra. Specifically, one can regard $\hat{A}$ as a (torsion-free) algebra over the ring of formal series $\mathbb{C}[[\hbar]]$, and recover the classical algebra of observables A as $\hat{A}/(\hbar)$. Then the Poisson bracket in A is given by $\{a,b\} = \lim_{\hbar \to 0} \frac{i}{\hbar}[\hat{a}, \hat{b}]$ where a, b are the limits of $\hat{a}$, $\hat{b}$ as $\hbar \to 0$. In such a situation, one says that the noncommutative algebra $\hat{A}$ is a *deformation quantization* of A, and A is called the *quasiclassical limit* of $\hat{A}$. Note also that under the above limiting procedure, the Schrödinger equation degenerates into the Hamilton equation.

1.3 Poisson homology.

An important invariant of a Poisson algebra A is its zeroth Poisson homology $HP_0(A) := A/\{A, A\}$. The meaning of this space is very simple: a linear function on $HP_0(A)$ is just a linear function on A (i.e., a generalized function, or distribution, on the the space of states of the classical mechanical system) which is invariant under the Hamiltonian flow with any Hamiltonian function. This space is important in Poisson geometry, and is also relevant in the context of quantization, as $\dim HP_0(A)$, when finite, provides an upper bound for the number of irreducible finite dimensional representations of any quantization $\hat{A}$ of A. So it is interesting to study the structure of the space $HP_0(A)$ for various Poisson algebras A.

The space $HP_0(A)$ for Poisson algebras A over the field of complex numbers was studied by many authors, see e.g. [4]. In particular it is known that if M is a Poisson algebraic variety with finitely many symplectic leaves, then the space $HP_0(\mathcal{O}(M))$ is finite dimensional. For instance, if M is a surface in the 3-dimensional space defined by a quasihomogeneous equation $Q(x, y, z) = 0$ which has an isolated singularity at the origin, then $HP_0(\mathcal{O}(M))$ is isomorphic to the Jacobi ring of the singularity, $J(f) := \mathbb{C}[x, y, z]/(f_x, f_y, f_z)$, and its dimension is the Milnor number $\mu(f)$ of the singularity.

The main goal of this paper is to extend these results to the case of a field $\mathbb{F}$ of positive characteristic p. In this case, the space $HP_0(A)$, where $A = \mathcal{O}(M)$, is no longer finite dimensional, even if M has finitely many symplectic leaves. Rather, $HP_0(A)$ is a finitely generated module over A^p, the algebra of p-th powers of functions from A. If A is graded, this is a graded module, so in particular it has a Hilbert series $h(t)$, which by Hilbert's syzygies theorem is a rational function of t. Our goal is to compute this function for M being a surface in $\mathbb{F}^3$ with an isolated singularity.

1.4 Our results.

Our main results are as follows.

In Section 3 we compute the Hilbert series of $HP_0(\mathcal{O}(M))$ for M being a Kleinian surface corresponding to a simple singularity of type A, D, E. On the basis of computer calculations, we conjecture a general formula for the answer, expected to hold if $p > h$, where h is the Coxeter number. We also prove our formulas for types A and D (i.e., for cyclic groups Cyc_n and dicyclic groups Dic_n).

In Section 4 we conjecture a general formula for the Hilbert series of $HP_0(\mathcal{O}(M))$, where M is defined by a homogeneous polynomial equation $Q(x, y, z) = 0$ of degree d, which is supposed to hold if $p > 3d - 6$. We also give conjectural formulas for several quasihomogeneous polynomials, based on computer calculations.

2 Preliminaries.

2.1 Poisson Algebras.

Definition 2.1. A Poisson algebra $(A, \{-,-\})$ is a commutative algebra A, equipped with a bilinear operation $\{-,-\}: A \times A \to A$, called a Poisson bracket, which satisfies the following properties:

- Skew-symmetry: $\{x, y\} = -\{y, x\}$
- Jacobi Identity: $\{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = 0$
- Leibniz Rule: $\{x, yz\} = y\{x, z\} + z\{x, y\}$

Note that $(A, \{-,-\})$ is a Lie algebra.

Example 2.1. Let $A = \mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We can define a Poisson bracket on A in the following manner: $\{x_i, x_j\} = \{y_i, y_j\} = 0, \ \{y_i, x_j\} = \delta_{ij}$.

Definition 2.2. Let A be a Poisson algebra. An ideal I of A is a *Poisson ideal* if $\{a, b\} \in I$ for all $a \in A, b \in I$.

Definition 2.3. A linear action of a group G on a Poisson algebra A is called a *Poisson action* if we have $g(ab) = (ga)(gb)$ and $g\{a, b\} = \{ga, gb\}$ for all $a, b \in A, g \in G$.

Definition 2.4. If A carries an action of G we let $A^G = \{a \in A | ga = a\}$ denote the algebra of G-invariants of A.

2.1.1 The zeroth Poisson homology $HP_0(A)$.

Definition 2.5. We let $\{A, A\} = \text{span}\{\{a_1, a_2\} \mid a_1, a_2 \in A\}$.

Definition 2.6. The zeroth Poisson homology $HP_0(A)$ of A is defined as the vector space $HP_0(A) = A/\{A, A\}$.

2.1.2 Interpretation of $HP_0(A)^*$ as the space of Poisson traces.

Let A be a Poisson algebra over a field $\mathbb{F}$. Linear functionals $A \to \mathbb{F}$ satisfying $\{a, b\} \mapsto 0$ are called *Poisson traces* on A. The space of Poisson traces $HP_0(A)^*$ is dual to the zeroth Poisson homology $HP_0(A)$.

2.1.3 $A/\{A, A\}$ as a module over A^p.

Theorem 2.1. *If A is a Poisson algebra over a field of characteristic p where p is prime, then $A/\{A, A\}$ is a module over A^p. If A is a finitely generated algebra, then $A/\{A, A\}$ is a finitely generated module over A^p.*

Proof. Consider $f^p \in A^p$. Then since $\{f^p, b\} = pf^{p-1}\{f, b\} = 0$, we have $f^p\{a, b\} = \{f^p a, b\}$ by the Leibniz rule, which implies the first statement. To prove the second statement note that the algebra A^p is Noetherian, since so is A, and $A^p \cong A$ with scalar multiplication twisted by $\lambda \mapsto \lambda^p$. Also, A is a finitely-generated module over A^p : if $f_1, ..., f_n$ are generators of A as an algebra, then generators of A as a module over A^p are $f_1^{i_1} ... f_n^{i_n}$, where $0 \le i_1, ..., i_n \le p-1$. Thus it immediately follows that the A^p-module $A/\{A, A\}$ of A is also finitely generated (with the same generators). □

Definition 2.7. Let A be an associative algebra over a field $\mathbb{F}$. Define the *zeroth Hochschild homology* to be $HH_0(A) = A/[A, A]$ and $HH_0^*(A) = \{\phi : A \to \mathbb{F} \mid \phi([a, b]) = 0 \; \forall a, b \in A\}$ to be the space of all *Hochschild traces*.

2.1.4 $G = \mathrm{Cyc}_n$ and Dic_n.

Definition 2.8. The cyclic group Cyc_n of order n and the dicyclic group Dic_n of order $4n$ are given by the following presentations:

$$\mathrm{Cyc}_n =< g | g^n = 1 >, \qquad \mathrm{Dic}_n =< a, b | a^n = b^2, b^4 = 1, b^{-1}ab = a^{-1} >$$

The groups Cyc_n and Dic_n each act on $\mathbb{F}^2$ as follows: $g \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$ where ω is an nth primitive root of unity, and $a \mapsto \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_1^{-1} \end{bmatrix}, b \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ where ω_1 is a $2n$th primitive root of unity (we assume that the characteristic of $\mathbb{F}$ does not divide the order of the group).

Remark 2.1. In the Cyc_n case, the action induces an action on $\mathbb{F}[x, y]$ as follows: $g \circ x = \omega x, g \circ y = \omega^{-1} y$. In the Dic_n case, we have the following induced action: $a \circ x = \omega_1 x,\ a \circ y = \omega_1^{-1} y,\ b \circ x = -y,\ b \circ y = x$.

2.2 Affine Poisson varieties and symplectic leaves

While we will primarily be concerned with Poisson algebras, it is useful to introduce the language of algebraic geometry. The geometric counterpart of a Poisson algebra is an affine Poisson variety:

Definition 2.9. An affine algebraic variety X is *Poisson* if $\mathcal{O}_X$ is a finitely generated Poisson algebra, i.e, an affine Poisson variety is of the form $\mathrm{Spec}(\mathcal{O}_X)$ where $\mathcal{O}_X$ is a finitely generated Poisson algebra without nonzero nilpotent elements.

Recall that a morphism $\phi : X \to Y$ between affine varieties X, Y by definition corresponds to a homomorphism $\phi^{\#} : \mathcal{O}_Y \to \mathcal{O}_X$. The map $\phi^{\#}$ is called the comorphism attached to ϕ.

Definition 2.10. A morphism $\phi : X \to Y$ between two Poisson varieties X and Y is *Poisson* if its corresponding comorphism is a Poisson homomorphism.

Definition 2.11. A *Poisson subvariety* of an affine Poisson variety X is an affine subvariety Y such that the inclusion $i : Y \to X$ is Poisson.

Definition 2.12. Given a Poisson variety X and $f \in \mathcal{O}_X$, we define the *Hamiltonian vector field* χ_f of f to be $\chi_f = \{f, -\} : \mathcal{O}_X \to \mathcal{O}_X, g \mapsto \{f, g\}$.

Definition 2.13. We define $\mathrm{Vect}(X)$ to be the vector space of all vector fields of a variety X and $\mathrm{HVect}(X) \subseteq \mathrm{Vect}(X)$ to be the vector space of all Hamiltonian vector fields.

Definition 2.14. Let X be a smooth Poisson variety, and T_pX be the tangent space at some $p \in X$. For a vector field χ on X, and $p \in X$, let $\chi|_p \in T_pX$ denote its value at p. Denote

$$HT_pX = \mathrm{span}\{\chi|_p \in T_pX \mid \chi \in \mathrm{HVect}(X)\}.$$

A Poisson variety X is called *symplectic* if $HT_pX = T_pX$ for all $p \in X$.

Definition 2.15. A Poisson variety X has *finitely many symplectic leaves* if we have a decomposition $X = \bigsqcup_{i=1}^{n} L_i$ into locally closed symplectic Poisson subvarieties $L_1, L_2, ..., L_n$. Each L_i is referred to as a *symplectic leaf.*

Example 2.2. [1] Let V be a symplectic vector space (i.e, a vector space equipped with a symplectic form ω) and $G \subseteq Sp(V)$ be a finite group acting on V that preserves the symplectic form. Then let $\pi : V \to V/G$ be the orbit map, and for a subgroup $H \leq G$ let $V_{(H)} = \{v \in V \mid aG_va^{-1} = H \text{ for some } a \in G\}$ where G_v is the stabilizer subgroup of v in G. Then, $V/G = \bigsqcup_{H \leq G} \pi(V_{(H)})$ where $\pi(V_{(H)})$ are its symplectic leaves. (The union runs over conjugacy classes of subgroups.) Note $\pi(V_{(H)})$ may be empty for a particular H, because it could happen that H does not occur as a stabilizer for any point.

2.2.1 Deformation quantizations.

Definition 2.16. Let A_0 be a commutative algebra over a field $\mathbb{F}$. Then we define the power series module $A_0[[\hbar]]$ by $A_0[[\hbar]] = \{\sum_{m \geq 0} a_m \hbar^m | a_m \in A_0\}$. Note that it is allowed for infinitely many a_m to be nonzero.

Definition 2.17. A *deformation quantization* $A_\hbar$ of a commutative algebra A_0 is the structure of a $\mathbb{F}[[\hbar]]$-algebra on $A_0[[\hbar]]$, i.e. a $\mathbb{F}[[\hbar]]$-bilinear map: $* : A_0[[\hbar]] \times A_0[[\hbar]] \to A_0[[\hbar]]$, $a \times b \mapsto a * b$, such that $a * b = ab \mod \hbar$. Note that $A_\hbar / \hbar A_\hbar \cong A_0$ as an algebra.

Definition 2.18. A *deformation quantization* $A_\hbar$ of a Poisson algebra A_0 is a deformation quantization of the commutative algebra A_0 such that, for all $a, b \in A_0$, we have: $a*b-b*a = \{a,b\}\hbar \mod \hbar^2$. Note that the Poisson bracket is uniquely determined by the product on $A_\hbar$, but not vice versa. Therefore a Poisson algebra may be considered a first-order deformation of the commutative algebra structure on A_0.

2.3 Known Results in Characteristic 0.

2.3.1 $A/\{A, A\}$ is finite-dimensional if Spec(A) has finitely many symplectic leaves

Theorem 2.2. *[4] Let X be an affine Poisson variety, Y be another affine variety, and $\phi : X \to Y$ be a morphism. If X has finitely many symplectic leaves and ϕ is finite, then $\mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_Y\}$ is finite-dimensional.*

Corollary 1. *Given a Poisson algebra A, if Spec(A) has finitely many symplectic leaves, then $HP_0(A)$ is finite-dimensional.*

Corollary 2. *Let X be an affine Poisson variety with finitely many symplectic leaves and G be a finite group acting on X. Then, $\mathcal{O}_X/\{\mathcal{O}_{X/G}, \mathcal{O}_X\}$ is finite dimensional. In particular, the subspace of G-invariants, $HP_0(\mathcal{O}_{X/G})$ is finite dimensional.*

Theorem 2.3. *[4] Let A be a nonnegatively filtered associative algebra, such that $A_0 = gr(A)$ is a finitely generated module over its center Z, and let $X =$ Spec(Z) be the corresponding Poisson variety. Assume that X has finitely many symplectic leaves. Then the space $A/[A, A]$ is finite-dimensional, and A has finitely many irreducible representations.*

The previous theorem extends to deformation quantizations of A_0, by replacing A with A_0's deformation, $A_\hbar$.

Theorem 2.4. *[4] Suppose that A_0 is a finitely generated module over its center Z, and Spec(Z) is a Poisson variety with finitely many symplectic leaves. Then $HH_0(A_\hbar[\hbar^{-1}])$ is finite-dimensional over $\mathbb{F}((\hbar))$, and $A_\hbar[\hbar^{-1}]$ has finitely many irreducible representations.*

3 Computing $HP_0(A)$ for $A = \mathcal{O}(V)^G$ and dim $V = 2$.

Let $V = \mathbb{F}^2$ as a vector space. In this section we let $A = \mathcal{O}(V)^G$ for some group G acting linearly on V and compute the Hilbert-Poincaré series for $HP_0(A)$.

3.1 Definitions.

Definition 3.1. Consider $\mathcal{O}(V) \cong \mathbb{F}[x_1, y_1]$. Then $\mathcal{O}(V)$ is a Poisson algebra as defined in Example 2.1.

Definition 3.2. We form a decomposition on $HP_0(A) = A/\{A, A\} = \bigoplus_{d \geq 0}(A_d/\{A, A\}_d)$ into finite-dimensional subspaces $A_d/\{A, A\}_d$ consisting of homogeneous polynomials of degree d.

Definition 3.3. Let V be a vector space over a field $\mathbb{F}$ and $V = \bigoplus_{n \geq 0} V_i$ be a decomposition of V into finite-dimensional subspaces V_i. The *Hilbert-Poincaré series* (or Hilbert series) of V is defined to be the formal power series $h(V; t) = \sum_{i \geq 0} \dim V_i \, t^i$.

In this paper we in particular calculate $h(HP_0; t) = \displaystyle\sum_{d \geq 0} \dim A_d \, t^d$.

Definition 3.4. Let B be a commutative algebra. We say that B has *primary generators* $z_1, ..., z_n$ and *secondary generators* $w_1, ..., w_m$ if $\{z_1^{k_1} ... z_n^{k_n} w_i \mid k_1, ..., k_n \in \mathbb{Z}_{\geq 0}, i \in \{1, ..., n\}\}$ is a basis of B. Note that primary generators and secondary generators do not necessarily exist for a general commutative algebra B; they exist if and only if B is a Cohen-Macaulay algebra.

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3.2 The results for the cyclic and dicyclic case.

We first calculate the invariants of A under G and then calculate $h(HP_0; t)$.

Proposition 3.1. *Assume that we are working over a field of characteristic* 0 *or* p *such that* $p \nmid n$*. The generating invariants of* A *under the action of* $G = \mathrm{Cyc}_n$ *are* $\{x^n, y^n, xy\}$*. Furthermore,* $A_{prim} = \{x^n, y^n\}$ *and* $A_{sec} = \{1, xy, (xy)^2, ..., (xy)^{n-1}\}$ *can serve as the primary and secondary generators of* A*.*

Proof. The proof is well known, but we include it for the reader's convenience. Consider $p(x) = \sum_{i=0}^{m} k_i x^{a_i} y^{b_i} \in A$ where $k_i \in \mathbb{F}, m, a_i, b_i \in \mathbb{Z}_{\geq 0}$. It's clear that this is invariant under g if and only if $n \mid (a_i - b_i)$ for all i. So it is sufficient to consider the case $p(x) = x^a y^b$ where $n \mid (a - b)$. Suppose $a \geq b$. Let $a - b = n\alpha$ for some $\alpha \in \mathbb{Z}_{\geq 0}$. Then, $x^a y^b = x^{n\alpha}(xy)^b = x^{n(\alpha + \lfloor \frac{b}{n} \rfloor)} y^{n \lfloor \frac{b}{n} \rfloor} (xy)^{b - n\lfloor \frac{b}{n} \rfloor}$. We can write this expression similarly if $b > a$. Thus, $A_{gen} = \{x^n, y^n, xy\}$ is a generating set for A, and furthermore, since $b - n\lfloor \frac{b}{n} \rfloor \in \{0, 1, ..., n-1\}$, $A_{prim} = \{x^n, y^n\}$ and $A_{sec} = \{1, xy, (xy)^2, ..., (xy)^{n-1}\}$ are the sets of primary and secondary generators of A. □

Theorem 3.2. *Let char* $\mathbb{F} = 0$*,* $G = \mathrm{Cyc}_n$*. Then* $h(HP_0; t) = \dfrac{1 - t^{2n-2}}{1 - t^2} = \sum_{m=0}^{n-2} t^{2m}$*.*

Proof. Recall that given a graded vector space V with a graded subspace W, we have $h(V/W; t) = h(V; t) - h(W; t)$. Thus, to compute $HP_0(A)$, we may compute $h(A; t)$ and $h(\{A, A\}; t)$ separately. We see from above that the basis for A will be terms of the form $x^a y^b$ with $n \mid a - b$ where $a, b \in \mathbb{Z}_{\geq 0}$. It follows that $h(A; t) = \frac{1}{(1-t)(1-t^n)}$.

We now compute a basis for $\{A, A\}$. Recall that $\{A, A\} = \mathrm{span}\{\{r, s\} | r \in A, s \in A_{gen}\}$. We have the formula $\{x^c y^d, x^a y^b\} = (ad - bc)x^{c+a-1} y^{d+b-1}$ and in particular

$$\{x^a y^b, xy\} = (b - a)x^a y^b, \{x^a y^b, x^n\} = bnx^{a+n-1} y^{b-1}, \{x^a y^b, y^n\} = any^{b+n-1} x^{a-1}$$

where $n | (b - a)$. Note that we obtain all monomials in A except those of the form $x^{a+c-1} y^{b+c-1}$ where $a, b = 0, c \in \{1, ..., n-1\}$ (since in that case $\{x^a y^b, (xy)^c\} = \{1, (xy)^c\} = 0$). Thus, $h(HP_0(A); t) = h(A; t) - h(\{A, A\}; t) = \dfrac{1 - t^{2n-2}}{1 - t^2} = \sum_{m=0}^{n-2} t^{2m}$.

□

Theorem 3.3. *Let char* $\mathbb{F} = p$ *where* $p > n$. *Then* $h(HP_0(A);t) = \dfrac{1-t^{2n-2}}{1-t^2} + \dfrac{t^{2p-2}(1+t^{np})}{(1-t^{2p})(1-t^{np})}$.

Proof. As above, $\{x^n, y^n, xy\}$ is a generating set for A, $h(A;t) = \dfrac{1}{(1-t)(1-t^n)}$ and

$$h(HP_0(A);t) = h(A;t) - h(\{A,A\};t)$$

Our calculation differs in finding a basis of $\{A,A\}$. As before,

$$\{A,A\} = \text{span}\{\{r,s\} | r \in A, s \in A_{gen}\}$$

and $\{x^a y^b, xy\} = (b-a)x^a y^b \qquad \{x^a y^b, x^n\} = bnx^{a+n-1}y^{b-1} \qquad \{x^a y^b, y^n\} = any^{b+n-1}x^{a-1}$ where $n|(b-a)$ and $c \in \{1, ..., n-1\}$. From the first equation, we obtain all monomials $x^k y^l$ for which $k - l \neq 0$ mod p. From the second, we obtain all monomials $x^k y^l$ with $k \geq n$, and $l + 1 \neq 0 \mod p$. From the third, we obtain all monomials $x^k y^l$ with $l \geq n$ and $k+1 \neq 0$ mod p. Thus, the contribution in degree less than or equal to n is $1+t^2+\cdots+t^{2(n-2)} = \frac{1-t^{2n-2}}{1-t^2}$.

In degree greater than n, we have monomials $(xy)^a x^{bnp}$ and $(xy)^a y^{cnp}$, where $c \geq 1$, and we require $a = -1 \mod p$. We have Hilbert series: $\frac{t^{2p-2}}{1-t^{2p}}$, $\frac{1}{1-t^{np}}$, $\frac{t^{np}}{1-t^{np}}$, enumerating the contribution from $(xy)^a$, x^{bnp}, y^{cnp}, respectively. The formula in the theorem follows. $\square$

Proposition 3.4. *Assume we work over characteristic* 0 *or* p *coprime with* $2n$. *The algebraic generators for the invariants of* A *under the action of* $G = \text{Dic}_n$ *are*

$$A_{gen} = \{x^{2n} + y^{2n}, xy(x^{2n} - y^{2n}), x^2y^2\}$$

Furthermore, $A_{prim} = \{(x^{2n} + y^{2n}), xy(x^{2n} - y^{2n})\}$ *and* $A_{sec} = \{1, x^2y^2, (x^2y^2)^2, ..., (x^2y^2)^n\}$ *can serve as sets of primary and secondary generators of* A.

As this is well known, we will omit the proof as it is similar to the proof in the type Cyc_n case.

Theorem 3.5. *Let $G = \mathrm{Dic}_n$. Suppose $p > 2n+2$. Then we have:*

$$h(HP_0(A);t) = \sum_{i=0}^{n} t^{4i} + t^{2n} + t^{2p-2}\frac{1+t^{(2n+2)p}}{(1-t^{4p})(1-t^{2np})}.$$

Proof. First we compute a basis of invariants. We begin with monomials $x^k y^l$ invariant under $\mathrm{Cyc}_{2n} \subset \mathrm{Dic}_n$, i.e. satisfying $2n|(k-l)$. To find all invariants of $\mathbb{F}[x,y]$ under $G' =< b|b^4 = 1 >$, we apply the symmetrizing element $e = 1 + b + b^2 + b^3$ to a general monomial $a = x^k y^l$:

$$\begin{aligned} e \circ x^k y^l &= x^k y^l + (-1)^k y^k x^l + (-1)^{k+l} x^k y^l + (-1)^{2k+l} y^k x^l \\ &= (1 + (-1)^{k+l}) x^k y^l + ((-1)^k + (-1)^l) x^l y^k \end{aligned}$$

Note that this is zero if $k+l$ is odd. We let:

$$p_{k,l} := \frac{1}{2} e \circ x^k y^l = x^k y^l + (-1)^k x^k y^l.$$

Then $\{p_{k,l} | k \geq l, 2n|k-l\}$ forms a basis for A. Straightforward computation gives:

$$\{p_{k,l}, p_{m,n}\} = (lm - kn)p_{k+m-1,l+n-1} + (-1)^k (km - ln) p_{l+m-1,k+n-1}.$$

We thus compute $\{A, A\} = \{A, A_{gen}\}$ as follows:

$$\begin{aligned} \{p_{k,l}, p_{2,2}\} &= 4(l-k) p_{\{k+1,l+1\}} \\ \{p_{k,l}, p_{2n,0}\} &= 2nl p_{k+2n-1,l-1} + 2nk p_{k-1,l+2n-1} \\ \{p_{k,l}, p_{2n+1,1}\} &= ((2n+1)l - k) p_{2n+k,l} + ((2n+1)k - l) p_{\{k,2n+l\}} \end{aligned}$$

We see that any basis element of the form $p_{k+1,l+1}$ is zero, unless $l-k$ is divisible by p (and thus by $2np$), from the first relation. Thus we may assume $k = l \mod p$. Then from the second relation, we see that if $k - l > 2n$, $k \geq 2n - 1$, and $l + 1 \neq 0 \mod p$, then we can replace $p_{k,l}$ with $p_{\{k-2n,l+2n\}}$, thus reducing $|k - l|$. On the other hand, if $k - l \leq 2n < p$, then we must have $k = l$. In this case, the third relation tells us that:

$$0 = -2nl(p_{k,l} + p_{l,k}) = \begin{cases} 4nlp_{k,l}, & l \text{ even} \\ 0, l \text{ odd} \end{cases}$$

We are thus left with the following monomials: 1) $p_{l,l}$, for $2 \leq 2n - 2$, 1, and $p_{n,n}$; 2) $p_{(2nk+l)p-1,pl-1}$, for $k, l \geq 1$; and 3) $p_{2kp-1,2kp-1}$, for $k \geq 1$.

It is straightforward to compute the Hilbert series for each type of monomial. We find:

$$HS = HS_0 + \frac{t^{2np}t^{2p-2}}{(1 - t^{2p})(1 - t^{2np})} + \frac{t^{2p-2}}{1 - t^{4p}} = HS_0 + t^{2p-2}\frac{1 + t^{(2n+2)p}}{(1 - t^{4p})(1 - t^{2np})},$$

where HS_0 is the Hilbert series in characteristic zero. □

3.3 Conjectures for a general Kleinian group

Recall that finite Kleinian groups (i.e., subgroups of $SL_2(\mathbb{C})$) are Cyc_n, Dic_n, T, C, and I, where T, C, I are the double covers of the groups of rotations of the regular tetrahedron, cube, and icosahedron. Via McKay's correspondence, they correspond to Dynkin diagrams (or roots systems) A_{n-1}, D_{n+2}, E_6, E_7, and E_8. Note that the orders of T, C, I are $24, 48$ and 120, respectively.

Definition 3.5. Let G be a finite subgroup of $SL_2(\mathbb{C})$. The *exponents* $m_1 \leq m_2 \leq ... \leq m_r$ of G are the exponents of the root system (or Weyl group) corresponding to the group G by the McKay correspondence.

Thus, the exponents m_i for Cyc_n are $1, 2..., n-1$, for Dic_n are $1, 3, ..., 2n+1$ and $n+1$ (so for even n there is a repeated exponent), for T are $1, 4, 5, 7, 8, 11$, for C are $1, 5, 7, 9, 11, 13, 17$, and for I are $1, 7, 11, 13, 17, 19, 23, 29$. (see [6]).

Definition 3.6. Let G be a finite subgroup of $SL_2(\mathbb{C})$. The *Coxeter number* h of G is defined by $h = 1 + m_r$, where m_r is its largest exponent.

Thus, the Coxeter number h for Cyc_n is n, for Dic_n is $2n + 2$, for T is 12, for C is 18, and for I is 30.

Theorem 3.6. *[3] Let G be a finite Kleinian group with exponents m_i and $A = \mathbb{C}[x, y]^G$. Then $h(HP_0; t) = \sum_{i=1}^{r} t^{2(m_i-1)}$.*

Now consider the situation in positive characteristic p. Note that if p does not divide the order of a finite Kleinian group G then G embeds into $SL_2(\mathbb{F})$.

It is well known that $\mathbb{C}[x, y]^G$ can be generated by three elements X, Y, Z of degrees $a, b, a + b - 2 = h$. The degrees a, b for Cyc_n are $2, n$, for Dic_n are $4, 2n$, for T are $6, 8$, for C are $8, 12$, and for I are $12, 20$. We state the following conjecture based on data from computations done in MAGMA.

Conjecture 3.1. Let char $\mathbb{F} = p$, G be a finite Kleinian group with exponents m_i and Coxeter number h. Let $p > h$ and $A = \mathbb{F}[x, y]^G$. Then

$$h(HP_0; t) = \sum_{i=1}^{r} t^{2(m_i-1)} + t^{2(p-1)} \frac{1 + t^{ph}}{(1 - t^{pa})(1 - t^{pb})}.$$

Note that we have proved this conjecture for the cyclic and dicyclic cases, so the only remaining cases are the exceptional groups T, C, I.

4 Poisson homology on cones of smooth projective and weighted projective curves

For each $(a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$, we define a grading on $\mathbb{F}_p[x, y, z]$ by declaring $\deg(x^k y^l z^m) = ak + bl + cm$. Let $Q(x, y, z)$ be a homogeneous polynomial of degree d in $\mathbb{F}_p[x, y, z]$ under this grading, and suppose that the surface $Q = 0$ is singular only at the point $(0,0,0)$. We define the Poisson bracket $\{,\}$ on $\mathbb{F}[x,y,z]/\langle Q\rangle$ as follows: $\{x,y\} = \frac{\partial Q}{\partial z}$, $\{y,z\} = \frac{\partial Q}{\partial x}$, $\{z,x\} = \frac{\partial Q}{\partial y}$.

4.1 The homogeneous (projective) case

Based on computer experimentation, we formulate the following conjecture:

Conjecture 4.1. Let $(a,b,c) = (1,1,1)$. Let $g = (d-1)(d-2)/2$, the genus of the projective curve $Q = 0$. Then for large enough p,

$$HS(t; HP_0(\mathbb{F}[x,y,z]/\langle Q\rangle)) = \frac{(1-t^{d-1})^3}{(1-t)^3} + t^{d-3}\phi(t^p),$$

where $\phi(z) := \frac{2gz}{1-z} + \frac{z^2+\sum_{j=1}^{d-1} z^j}{(1-z)^2}$.

The conjecture was verified by computer for $Q = x^d + y^d + z^d$ in the cases $(d,p) = (3,7)$, $(4,11)$, $(5,11)$, and $(7,17)$, up to degree 100. For $d = 3$ (where $\phi(z) = \frac{3z}{(1-z)^2}$), we can prove the conjecture, and we are working on a general proof.

4.2 The quasihomogeneous (weighted projective) case

More generally, we can consider the quasihomogeneous setting. Namely, let $A \geq B \geq C > 1$ be positive integers, and $d = LCM(A,B,C)$. Let $a = d/A, b = d/B, c = d/C$. Then we have a quasihomogeneous polynomial $Q(x,y,z) = x^A + y^B + z^C$, which has degree d, where x,y,z have degrees a,b,c (which are relatively prime). Let $h_{A,B,C,p}(t)$ be the Hilbert series of HP_0 of the surface X given by $Q(x,y,z) = 0$ over a field of characteristic p.

Proposition 4.1. *(see [5] and references therein) One has* $h_{A,B,C,0}(t) = \frac{(1-t^{d-a})(1-t^{d-b})(1-t^{d-c})}{(1-t^a)(1-t^b)(1-t^c)}$.

On the basis of computer calculations, we make the following conjecture:

Conjecture 4.2. For large p, $h_{A,B,C,p}(t) = h_{A,B,C,0}(t) + t^{d-a-b-c}\phi_{A,B,C}(t^p)$, where $\phi_{A,B,C}$ is a certain rational function independent on p.

Moreover, we expect that the answer is the same for any quasihomogeneous isolated singularity with the same degrees a, b, c, d.

Remark 4.1. In the homogeneous case $A = B = C = d$, the function $\phi_{A,B,C}$ is given by Conjecture 4.1.

Here are some conjectural formulas for $\phi_{A,B,C}$ in other (non-Kleinian) cases:

$$\phi_{6,3,2} = \frac{z}{(1-z)^2}; \ \phi_{8,4,2} = z + \frac{2z}{(1-z)(1-z^2)}; \ \phi_{9,3,3} = z + z^2 + \frac{3z}{(1-z)(1-z^3)}. \tag{1}$$

5 Conclusion

In this paper we have studied the behavior of the zeroth Poisson homology in characteristic p for Poisson algebras arising from surfaces with an isolated singularity, defined by a quasi-homogeneous polynomial equation $Q(x, y, z) = 0$. In characteristic zero, it is known that the zeroth Poisson homology HP_0 is isomorphic to the Jacobi ring of the singularity, and its dimension is the Milnor number. In positive characteristic, the HP_0 is infinite dimensional, but has a grading with finite dimensional homogeneous parts, so while we cannot talk about its dimension, we can consider its Hilbert series, and our goal is to compute this series. It turns out in examples that this series equals the Hilbert polynomial of the Jacobi ring plus a fixed power of t multiplied by a certain rational function of t^p, which is to be computed in each case.

The simplest surfaces with isolated singularity are Kleinian surfaces, which can also be obtained as quotients of the 2-dimensional space by the action of a finite subgroup Γ of $SL(2)$. We have found a conjectural formula for the Hilbert series in the case when p is bigger than the Coxeter number of Γ, and proved it for the groups Γ of type A and D.

We have also considered the case when $Q(x, y, z)$ is a homogeneous polynomial of degree d, and found a conjectural formula for the Hilbert series in this case. We are able to prove this formula if $Q(x, y, z) = x^3 + y^3 + z^3$.

6 Future research

Besides proof of the above conjectures, here are some directions of future research that would be interesting to explore.

1. Extend Conjecture 4.1 to quasihomogeneous polynomial equations.

2. Extend Conjecture 4.1 to small primes p.

3. Extend the results of this paper from surfaces to higher dimensional Poisson varieties, e.g. V/G, where V is a symplectic vector space and G a finite group acting linearly on V.

4. Show that if M is an affine Poisson algebraic variety over $\overline{\mathbb{Q}}$ with finitely many symplectic leaves, and M_p is the reduction of M modulo a large enough prime p, then the minimal number of generators of $HP_0(\mathcal{O}(M_p))$ over $\mathcal{O}(M_p)^p$ is bounded by a constant independent of p (this would be the characteristic p analog of the finite dimensionality of HP_0 in the case of finitely many symplectic leaves).

5. Let V be a finite dimensional symplectic vector space, and G acts on V preserving the symplectic structure (all in characteristic zero). Let $A = (SymV)^G$, and B be the reduction of A modulo p. Give an exact formula (for large p) for the Hilbert series of $HP_0(B)$ in terms of the zeroth Poisson homology of V_x/G_x in characteristic zero, where G_x is a stabilizer of a point $x \in V$, and $V_x = V/V^{G_x}$.

6. Extend the Hilbert series formula for Kleinian groups to the case $p \leq h$.

7. Find the Hilbert series formula in characteristic p for the zeroth Poisson homology of nilpotent cones of a simple of Lie algebras and, more generally, of their Slodowy slices (i.e., of classical finite W-algebras).

8. Find the Hilbert series formula in characteristic p for the zeroth Poisson homology of symmetric powers of quasihomogeneous isolated surface singularities.

9. Compute the Hilbert series for HP_0 in characteristic p for homogeneous (and quasihomogeneous) examples of the form $f = 0, g = 0$, where f, g are polynomials in four variables x, y, z, w, with $\{x, y\} = \det \partial(f, g)/\partial(z, w)$, etc. More generally, what happens for a complete intersection 2-dimensional surface in $\mathbb{A}^n$?

7 Results obtained after this paper was finished

After this paper was finished, significant progress on the problems posed in this paper was achieved in [2]. In particular, Conjectures 3.1,4.2, 4.1 were proved, as well as formulas (1), and moreover a general formula for the Hilbert series was found in the quasihomogeneous case. Also, problems 1,3,6 from the section "Future research" were solved, and partial progress was achieved in some of the other problems.

References

[1] K.A. Brown and I. Gordon. *Poisson Orders, Symplectic Reflection Algebras, and Representation Theory.* J. Reine Angew. 559 (2003), 193-216.

[2] Yongyi Chen, Pavel Etingof, David Jordan, Michael Zhang, Poisson traces in positive characteristic, arXiv:1112.6385.

[3] P. Etingof, S. Gong, A Pacchiano, Q. Ren, and T. Schedler. *Computational approaches to Poisson traces associated to finite subgroups of* $Sp_{2n}(\mathbb{C})$. arXiv:1101.5171v1 [math.SG]. 26 Jan 2011.

[4] P. Etingof and T. Schedler. *Poisson traces and D-Modules on Poisson varieties.* arXiv:0908.3868v4 [math.SG]. 22 Dec 2011.

[5] P. Etingof and T. Schedler. *Zeroth Poisson homology of symmetric powers of isolated quasihomogeneous surface singularities.* arXiv:0907.1715v1 [math.SG]. 10 July 2009.

[6] J. E. Humphreys, Reflection groups and Coxeter groups, Cambride University Press, 1990.