A Brief Survey on Wallpaper Groups

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1 Introduction

As the algebraic study of symmetries, group theory includes groups, subgroups, cosets, and much more. In this paper, we explore wallpaper groups, which is a part of group theory. We begin by discussing the basic terminology and transformations used in wallpaper groups, including isometries and fundamental regions. Next, we look over the existing combinations of wallpaper groups through some of their transformations. We proceed to introduce the Orbifold Notation and the "Cost" of groups with examples. Finally, we discuss some theorems, including the Magic Theorem, and branch out to related groups. As a topic, wallpaper groups are generally less focused on proofs and more focused on visuals and symmetric beauty. There are many artists, including Maurits Cornelis Escher, who are fascinated by these concepts of wallpaper groups and spend a lot of their time creating diagrams that represent the symmetries and transformations of these wallpaper groups.

2 Wallpaper Groups

Definition 2.1 A *binary operation* μ on a set G is a function

$$\mu : G \times G \to G.$$

Definition 2.2 A *group* is $(G, *)$ where G is a set and $*$ is a binary operation on G with the conditions:

(i) *Associativity:* For all elements a, b, $c \in G$, a group has to satisfy that

$$(a * b) * c = a * (b * c).$$

(ii) *Identity:* For all elements $a \in G$, there exists an *identity* element e in G such that

$$a * e = e * a = a.$$

(iii) *Inverse:* There exists an element $a^{-1} \in G$ such that

$$a * a^{-1} = a^{-1} * a = e$$

where all elements $a \in G$.

Before moving on, we can briefly look at some common groups that appear in group theory.

Example 1 A big category in groups are *cyclic groups*. A group G is *cylic* if there is some element $x \in G$ such that $G = \{x^n | n \in \mathbb{Z}\}$, when the operation is multiplication.

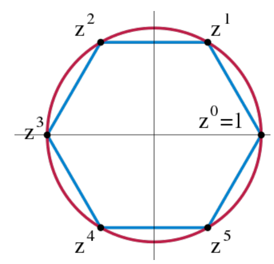

Figure 1. A cyclic group under multiplication.

Example 2 Another category in groups are *dihedral groups*. A *dihedral group* is a group of symmetries of a regular polygon.

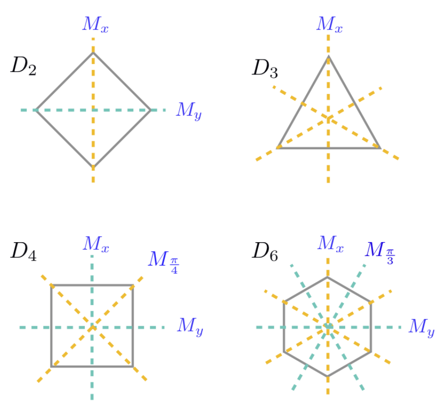

Figure 2. Dihedral groups for D_2, D_3, D_4, D_6

Definition 2.3 The *distance* $\mathrm{d}(x_1, y_1)$ in $\mathbb{R}^2$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where x and y are the coordinates of a point.

Definition 2.4 An *isometry* of n is a distance-preserving transformation of dimensional-space $\mathbb{R}^n$. For wallpaper groups, there are four possible isometries of the plane.

Definition 2.5 A *translation* is defined as $t_a : \mathbb{R}^2 \to \mathbb{R}^2$ and $x \mapsto x+a$ where $x \in X$. Notice that $\mathrm{d}(t_a(x), t_a(y)) = \mathrm{d}(x, y)$, meaning that t_a is distance-preserving.

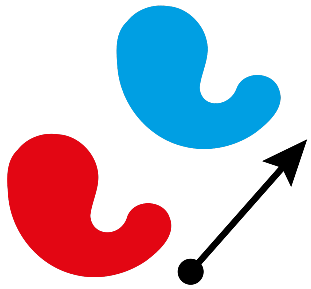

Figure 3. Example of a translation

Definition 2.6 A *reflection* over a line through the origin is defined as

$$Refl_l\ (v) = 2\frac{v \cdot l}{l \cdot l} l - v$$

where v denotes a vector, l denotes a vector in the direction of the line through the origin where the transformation is performed, and the $\cdot$ denotes the dot product. Note that a reflection is its own inverse and these lines are called mirror lines.

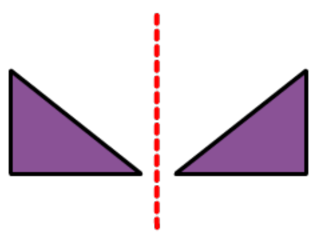

Figure 4. Example of a mirror line

Definition 2.7 A *rotation* is a change of angle around a center point. In terms of a matrix R, rotations are points that rotate counterclockwise around an angle θ where

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

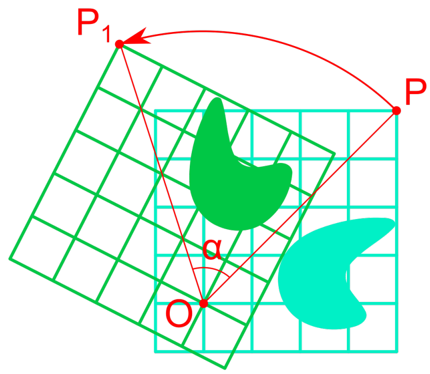

Figure 5. Example of a rotation

Definition 2.8 A *glide reflection* is a combined transformation of a reflection followed by a translation.

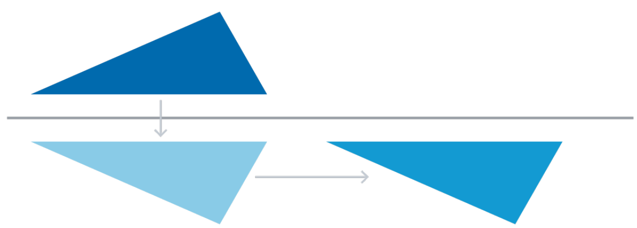

Figure 6. Example of a glide reflection

Remark 1 Geometrically, it is intuitive that reflections, rotations, and glide reflections are isometries. With some manipulations, it can also be shown algebraically. Since this paper does not cover these calculations, the readers can attempt this as an exercise.

Definition 2.9 A *fundemental region* is a region that can be infinitely repeated to fill up the $\mathbb{R}^2$ plane.

Definition 2.10 A *Wallpaper Group* is a plane of symmetries with a fundamental region and a combination of isometries being the elements.

A wallpaper group is a group because it fits into the conditions:

- *Associativity*: The elements of a wallpaper group are the different isometries. The order in which the transformations are performed does not matter. A glide reflection is the perfect example since reflecting and then translating is the same as translating and then reflecting.

- *Identity*: For a wallpaper group, since the elements are transformations, the identity is just the group itself with no transformations.

- *Inverse*: All transformations have inverses. Wallpaper groups are made of their transformations, so they naturally have inverses as well.

Definition 2.11 *Orbifold Notation* is a way to represent these transformations.

- First write down an integer n for a gyration point—rotation point without reflections—of order n. Refer to **Figure 7** for an example of 333.

- Next, write down a * for any mirror lines.

- On the right of the *, write down an integer m for a transformation of $2m$ that includes a reflection line on a rotation point. **Figure 8** shows an example of *2222.

- If there is a glide reflection, put x. **Figure 9** has an example of a glide reflection XX.

- If more than one mirror line doesn't intersect, put down * for each line. However, if the mirror lines do intersect, only put down one *. **Figure 10** is ** with two parallel mirror lines.

- Write down an O if there are no other symmetries outside of translations.

 
 

Figure 7. Notation 333 contains 3 gyration points
Figure 8. Notation *2222 contains 2 mirror lines and 4 rotation points
Figure 9. Notation XX contains 2 glide reflections
Figure 10. Notation ** contains 2 parallel mirror lines

Definition 2.12 For different transformations, there are different costs. We will use costs later in the theorems.

- An O costs 2
- A glide reflection x costs 1
- A * costs 1
- A gyration point n costs $\frac{n-1}{n}$
- A point n on the right of the * costs $\frac{n-1}{2n}$

3 Theorems

Theorem 1 The Magic Theorem *Every wallpaper group's total cost should be 2.*

Proof. This proof can be found in [1]. As the proof includes many concepts related to spherical maps, the *Euler Characteristic*, and other ideas outside the scope of this paper, we will not be including this proof. For more information, readers can go to Chapter 7 of [1].

Theorem 2 *There are exactly 17 wallpaper groups.*

Proof. Using the costs from above and the Magic Theorem, we can show that there are only 17 wallpaper groups with casework.

(i) There are no mirror lines. This would include cases with only rotations, translations, and glide reflections: 2222, 333, 442, 632, and O, xx, 22x.

(ii) One mirror line with rotations and glide reflections: 22*, 4*2, 3*3, and *x.

(iii) There is more than one mirror line: **, *2222, *333, *442, *632, and 2*22.

Thus, in total there are $7 + 4 + 6 = 17$ types of wallpaper groups.

Theorem 3 *Without rotations, there are only four types of wallpaper groups.*

Proof. Following the same method as Theorem 3, we can find all of these through casework. Since there are no rotations, these can only contain O, *, and X. Using the Magic Theorem, we know the only four possible types are:

- Only translations: O
- Two parallel lines: **
- A reflection line and a glide reflection in the same direction: *x
- Two glide reflections: xx

Theorem 4 *If there are only rotations and no glide reflections or mirror lines, then*

$$\sum_{i=1}^{k} \frac{n_i - 1}{n_i} = 2.$$

where $n_1 n_2 n_3 ... n_k$ describes the rotations.

Proof. Without mirror lines and glide reflections implies that there are no $*$ or X in orbifold notation. In other words, we know that all the rotations are gyrations. Referring back to the cost of a gyration point, the formula makes sense intuitively. From this, we can also find the four possible groups: 2222, 442, 333, and 632.

4 Extension

Wallpaper groups are a geometric, visual representation part of group theory. Related to wallpaper groups, Escher's Tessellations are pretty well-known in the field. Some of the figures covered in this paper are part of Escher's collection. Branching out from wallpaper groups, there are other groups such as the Frieze groups, and hyperbolic geometry with related properties.

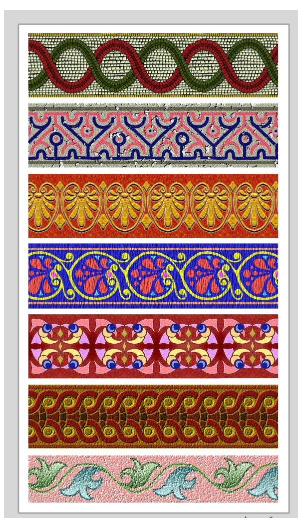

Figure 11. Example of Frieze groups
Figure 12. Example of Hyperbolic geometry

5 Acknowledgements

I would like to thank the MIT PRIMES Circle program, as well as our mentor, Gabrielle Kaili-May Liu. In addition, I would like to thank my parents for supporting me and driving me to MIT every week.

6 References

[1] J. H. Conway, H. Burgiel, C. Goodman-Stauss, *The Symmetries of Things*, Taylor & Francais Group, New York, 2008.
[2] P. J. Morandi, *The Classification of Wallpaper Patterns: From Group Cohomology to Escherís Tessellations*, New Mexico State University, April 2003.

7 Image References

[1] For figure 1 go to: `https://en.wikipedia.org/wiki/File:Cyclic_group.svg`
[2] For figure 2 go to: `https://www.google.com/url?sa=i&url=https%3A%2F%2Fwww.researchgate.net%2Ffigure%2FMirror-planes-of-the-dihedral-groups-D2-D3-D fig8_344734535&psig=AOvVaw0SF3zYxuaACI8-ub-qpxFw&ust=1684819948475000&source=images&cd=vfe&ved=0CBAQjRxqFwoTCJDSocaZiP8CFQAAAAAdAAAAABAJ`
[3] For figure 3 go to: `https://en.wikipedia.org/wiki/File:Traslazione_OK.svg`
[4] For figure 4 go to: `https://media.turtlediary.com/lesson/reflection-rotation-transla reflec-5.png`
[5] For figure 5 go to: `https://upload.wikimedia.org/wikipedia/commons/thumb/8/87/Rotation_illustration2.svg/1200px-Rotation_illustration2.svg.png`
[6] For figure 6 go to: `https://upload.wikimedia.org/wikipedia/commons/thumb/e/e8/Glide_reflection.svg/1200px-Glide_reflection.svg.png`
[7] For figure 7 go to: `https://theliteracyblog.com/2017/11/`
[8] For figure 8 go to: `https://www.pinterest.com/pin/64317100901032198/`
[9] For figure 9 go to: `https://artlandia.com/wonderland/glossary/WallpaperGroupPG.html`
[10] For figure 10 go to: `https://artlandia.com/wonderland/glossary/WallpaperGroupPM.html`

[11] For figure 11 go to: https://www.glassner.com/wp-content/uploads/2014/04/Creative-Art-Digital-Frieze-Groups.jpg
[12] For figure 12 go to: https://explorationofnon-euclideangeometries.weebly.com/uploads/1/0/9/2/109218259/haken-sized-figure03_orig.jpg