Graph Theory and the Optimization of City Infrastructure

Neil Jin and Apurva Varigonda

May 23, 2021

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1 Introduction to Graphs

1.1 Definitions

Contact tracing is a method which has become prevalent in the midst of the global 2019 COVID-19 pandemic. This method is being used by governments and healthcare systems globally to try and curb the case count for the pandemic. By seeing who infected people have been close to, it is easier for governments to quarantine them and hopefully stop the spread of the virus to more people. Often, the spread of the virus is represented using diagrams as in Figure 1. The vertices represent people in the network and the edges represent connections between people through which the virus can spread. The red represents infected individuals.

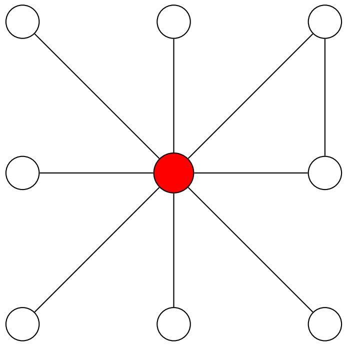

Figure 1. Contact Tracing Graph

While this is one use for a graph, there are many other uses as well. For example, graphs are also used to describe many algorithms in computer science such as in machine learning. Towards the end of this paper, we discuss the applications of graph theory to optimizing city infrastructure.

However, first we introduce a mathematically rigorous definition of a graph.

Definition 1.1. A *graph* G consists of a finite nonempty set V of objects called vertices and a set E of 2-element subsets of V called edges.

The two sets V and E are called the *vertex set* and *edge set* of G respectively. A vertex v is an object in a graph, and is commonly drawn as a point or circle. An edge e is a line that connects two vertices.

Definition 1.2. The number of vertices is called the *order* of the graph, usually denoted by n.

Definition 1.3. The number of edges is called the *size* of the graph, usually denoted by m.

First we present an introductory example of a graph. Consider a social situation in which each person Alice, Bob, Carl, Dan, and Ethan, know one another, but not everyone knows everyone else. People who do not know each other shake hands to greet each other. This situation can be modeled by a graph where vertices are people and edges are handshakes (see Fig. 2). Then note that the vertex set of this graph is

$$V(G) = \{\text{Alice}, \text{Bob}, \text{Carl}, \text{Dan}, \text{Ethan}\}$$

and the edge set is

$$E(G) = \{(\text{Alice,Carl}), (\text{Alice,Ethan}), (\text{Bob,Dan}), (\text{Carl,Dan}), (\text{Dan,Ethan})\}.$$

Now we present a few more definitions that discuss vertex to vertex relations.

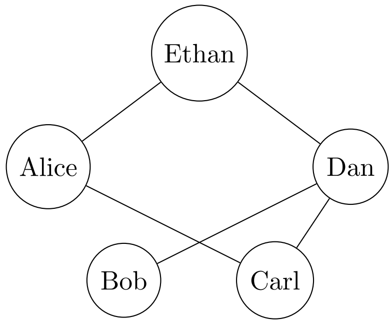

Figure 2. Social network modeled by a graph

Definition 1.4. If two vertices u and v are joined by an edge uv, then vertices u and v are said to be *adjacent*.

Remark 1.5. Additionally, the edge uv is said to be *incident* with vertex v. Similarly, uv is also incident with vertex u. Please refer to Figure 3.

Example. Figure 3 is an example of incident vertices.

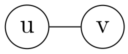

Figure 3. Incident and Adjacent

1.2 Graph Traversal

After constructing a graph, it is natural to consider how one might move around on a graph. The vertices can be a clear analogy for cities and the edges roads between cities, and traversing a graph provides a good model of traveling across these cities. In this vein, we define some terms that describe the different ways in which we can traverse a graph.

Definition 1.6. A *walk* is a sequence of vertices that are connected by edges in a graph.

Following this sequence, one can travel from a vertex v_1 to a vertex v_n. For example, a walk might be $\{v_1, v_2, \cdots, v_n\}$. In a walk, there can be repeated edges and vertices. A *trail* is similar to a walk, except that edges cannot be repeated.

Definition 1.7. A *path* is a sequence of vertices in a graph such that consecutive vertices in the sequence are adjacent and no vertices or edges are repeated.

Definition 1.8. A *circuit* is a trail that begins and ends on the same vertex.

Definition 1.9. A *cycle* is a circuit that does not repeat any vertices.

Example. Please refer to Figure 4.

- An example of a walk is $\{v_1, v_2, v_3, v_2\}$
- An example of a trail is $\{v_2, v_1, v_5, v_2, v_3\}$.

- An example of a path is $\{v_1, v_2, v_5, v_3\}$.
- An example of a circuit is $\{v_1, v_2, v_5, v_4, v_3, v_5, v_1\}$.
- An example of a cycle is $\{v_1, v_2, v_3, v_5, v_1\}$.

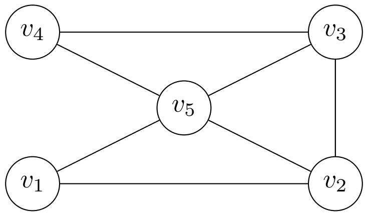

Figure 4. A generic graph

1.3 Connected Graphs

If there is a path from vertex v to vertex u, denoted by $u-v$, then u and v are said to be *connected.* A graph G is *connected* if every two vertices of G are connected. Two points are *disconnected* if there does not exist a path between them, and likewise, a graph is *disconnected* if there does not exist a path between every two vertices.

Definition 1.10. The *complement* of a graph G is denoted as $\bar{G}$. For every two vertices u and v in G, if there exists an edge uv in G, the edge uv does not exist in $\bar{G}$. Similarly, if the edge uv does not exist in G, $uv \in V(\bar{G})$.

Theorem 1.11. *If G is a connected graph of order 3 or more, then G contains two distinct vertices u and v such that $G-u$ and $G-v$ are connected.*

Proof. Let u and v be two distinct vertices of a graph G. Let $d(u,v) = \text{diam}(G)$. Suppose that $G-v$ is disconnected. Then $G-v$ contains two vertices x and y so that there does not exist an $x-y$ path. Since G itself is connected, there exists both a $u-x$ and $u-y$ path. Let P^x be a $u-x$ geodesic in G, and let P^y be a $u-y$ geodesic in G. Because $d_G(u,v) = \text{diam}(G)$, v cannot lie on P^x or P^y. The path P^x and then P^y produce an $x-y$ walk. If there exists a $x-y$ walk then there exists an $x-y$ path, and so $G-v$ is connected, which is a contradiction of the prior statement. □

Theorem 1.12. *If G is a disconnected graph, then $\bar{G}$ is connected.*

Proof. In a graph G, if for some two vertices u and v there does not exist a $u-v$ path, by definition in $\bar{G}$ there exists a $u-v$ path. If for some vertices u, v, and w in G where there exists a $u-v$ path but there does not exist a $u-w$ path or a $v-w$ path, in $\bar{G}$ there exists a $v-w$ path and a $u-w$ path. In $\bar{G}$ there then must exist a $u-v$ path $\{u, \cdots, w, \cdots, v\}$. Therefore, $\bar{G}$ is connected. □

1.4 Examples of Common Graphs

Here are some common categorizations of graphs:

- A graph G of size n which consists of a circuit is denoted as C_n
- A graph G of size n which consists of a cycle is denoted as K_n.
- A graph G of size n which consists of a path is denoted as P_n

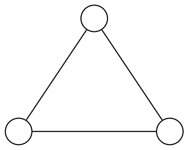

Figure 5. K_3

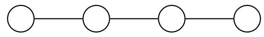

Figure 6. P_4

1.4.1 Graph Arithmetic

Definition 1.13. A *union* of two graphs G and H, denoted by $\cup$, is defined to be the graphs that has the union of the edge sets, $E(G) \cup E(H)$, as its edge set and the union of the vertex sets, $V(G) \cup V(H)$, as its vertex set.

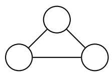

Figure 7. Graph G

Figure 8. Graph H

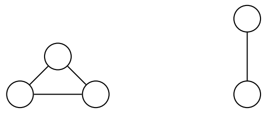

Figure 9. G ∪ H

Definition 1.14. To join two graphs G and H using *addition*, we connect each $v \in V(G)$ to each $v \in V(H)$. For example, the addition of P_2 and P_3 is below.

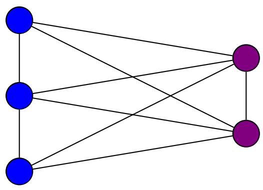

Figure 10. Graph Addition

Definition 1.15. The *Cartesian product* of two graphs G and H, denoted by $\times$ or $\square$, has the vertex set $V(G \times H) = V(G) \times V(H)$.

Example. If we have the two graphs P_3 and K_3, their Cartesian product would look like graph $P_3 \times K_3$ in Figure 13.

1.5 Degree

Definition 1.16. The *degree* of a vertex v is the number of edges that are incident with v. This is written as $\deg(v)$.

The first big theorem in graph theory tells us about the relationship between the degrees of the vertices in a graph and the size of the graph.

Theorem 1.17 (First Theorem of Graph Theory)**.** *In a graph G of size m,*

$$\sum_{v \in V(G)} deg(v) = 2m$$

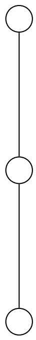

Figure 11. Graph P_3

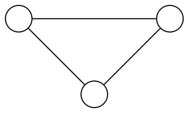

Figure 12. Graph K_3

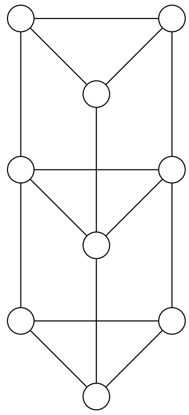

Figure 13. Graph $P_3 \times K_3$

Proof. When taking the sum of degrees of vertices $v \in V(G)$, each edge is connected to two vertices so it is counted twice. $\square$

1.6 Multigraphs, Weighted Graphs, and Digraphs

Definition 1.18. A *multigraph* is a graph that can have more than one edge connecting two distinct vertices.

Definition 1.19. A *weighted graph* is constructed by taking all edges in a multigraph that join the same two vertices and collapsing them. Then we assign each edge a whole number (the *weight* of the edge) which represents the number of edges between two vertices.

Definition 1.20. A *digraph* is a graph that has directed edges, or edges that can only be traversed in one direction.

1.6.1 Examples

Example. Figure 14 gives us an example of a digraph.

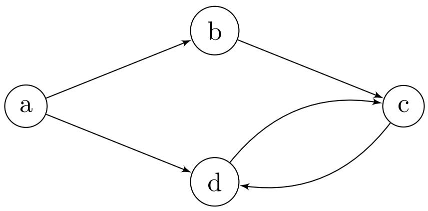

Figure 14. Digraph

Example. Figure 15 is an example of a multigraph.

Example. Lastly, we have an example of a weighted graph in Figure 16.

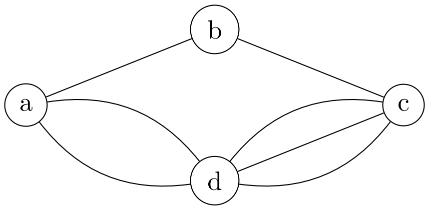

Figure 15. Multigraph

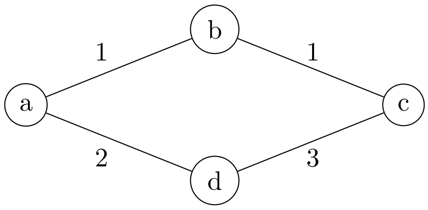

Figure 16. A weighted graph which corresponds to Figure 15

2 A Brief Interlude on Group Theory

2.1 Definition and Examples

Definition 2.1. Let G be a set under a binary operation. For that set to be a *group* it needs to satisfy 4 conditions:

1. The binary operation must be associative for all elements of G. This means that for all a, b, and c in G, $(ab)c = a(bc)$ must always be true.
2. The group must contain an identity element such that $ae = ea = a$. Here, e denotes the identity.
3. The group must contain an inverse b for each element a such that $ab = ba = e$.
4. The group must be closed. That is, for every two elements a and b in group G, ab must also be in the group G.

Definition 2.2. A subset of groups have the property of being *abelian*. This means that all of the elements of the group commute, or that $ab = ba$ for all $a, b \in G$.

Because groups need to satisfy so many conditions, its natural to think that they are not so common, but this is not the case!

Example. A simple example of a group is the set of all integers under the operation addition, denoted as $(\mathbb{Z}, +)$. Let us check that this group satisfies all of the group conditions:

1. It is associative since $a + (b + c) = (a + b) + c$.
2. It contains the identity element 0 because $a + 0 = 0 + a = a$.
3. It has an inverse because for all integers a, the inverse is $-a$.
4. It is closed since adding and subtracting integers still returns integers.

However, note that the set of all integers under multiplication $(\mathbb{Z}, \times)$ is not a group. It fails the third property for inverses because the element 5 doesn't have an inverse, i.e. there is not an integer such that $5 * a = 1$ in the group (since we only have integers).

2.2 Group Theorems

Theorem 2.3. *There is only one identity element in a group.*

Proof. We proceed by contradiction. Assume both e and e' are identity elements of a group. Therefore, $ae = a$ and $ae' = a$, so $ae = ae'$. Multiplying both sides by a^{-1} the inverse of a, we get $e = e'$ so there can only be one identity element. □

Theorem 2.4. *For every element a in a group G, there is only one inverse.*

Proof. We proceed by contradiction. Assume both b and c are inverses of a. This implies that both $ab = e$ and $ac = e$ are true. Therefore, $ab = ac$, and so we get $b = c$. □

Theorem 2.5. *For two elements a and b in a group G, $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. We start by multiplying $(ab)(b^{-1}a^{-1})$. Then we get $a(bb^{-1})a^{-1} = aea^{-1} = e$. Since $(ab)(ab)^{-1} = e$, we can see that $(ab)^{-1} = b^{-1}a^{-1}$. □

2.3 Subgroups and Cyclic Groups

Definition 2.6. H is a *subgroup* of G, denoted by $H \leq G$ if H is a subset of G and H is a group.

Example. Group G is the set of real numbers under addition, a subgroup of G, H, is the set of integers under addition.

Definition 2.7. The *center* of a group G, denoted by $Z(G)$, is the subset of elements in G that commutes with every element of G.

Definition 2.8. If a is an element in a group G, then the *centralizer* of a is the set of all elements that commute with a. In other words, the centralizer is a set such that $ab = ba$

Example. In the dihedral group which consists of the elements D_4, $V, H, R_0, R_{90}, R_{180}, R_{270}$ under function composition, only the identity element R_0 is the center of the group. However, the centralizer of a rotation R are all the possible rotations $R_0, R_{90}, R_{180}, R_{270}$. The centralizer of a reflection V or H are the two reflections and the identity rotation: V, H, R_0.

Definition 2.9. A group G is *cyclic* if there is an element a such that $G = \{a^n | n \in Z\}$. Here, a is called the *generator* of G.

Example. The set of integers under addition modulo m is a cyclic group. Here, 1 and -1 are generators.

Example. The set $\{i, -1, -i, 1\}$ where $i = \sqrt{-1}$ under multiplication is a cyclic group. Here, i is the generator.

2.4 Permutation Groups

Definition 2.10. A *permutation* of a set A is the function that maps the set back to itself. A *permutation group* is the set of permutations of A under the composition operation.

Example. For the set $\{1, 2, 3, 4\}$, a permutation could be

$$\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 4, \alpha(4) = 1.$$

In the context of group theory, we use a particular notation to represent permutations. A vertical representation is used where the map is from the top number in the stack to the bottom number in the stack. For example, if we had

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

this represents 1 being mapped to 2, 2 being mapped to 3, etc. We can also compose permutations. For example, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

However, this notation is often cumbersome so it is simplified using *cycle notation.* For example, if we have,

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

this permutation can be rewritten using cycle notation. Since 1 maps to 1, it gets its own parentheses, but is sometimes left out. We have that 2 maps to 4 and 4 maps back to 2, so we put these two values in their own set of parentheses. Next, 3 maps back to itself like 1, so it is given its own parentheses. Thus p can be rewritten as

$$p = (1)(24)(3).$$

We can compose permutation groups in cycle notation as well. For example, if we have (132)(4) and (1243) as our two permutations, we can compose them as (132)(4)(1243). However, this cycle notation had repeats in it, which is not very helpful. Therefore, we can also rewrite it in *disjoint* cycle notation where no element in the notation is written more than once.

3 Graph Relations

3.1 Isomorphisms

Can two graphs be the same? Let us look to Figures 17 and 18. Although these two graphs look quite different at first glance, both of these graphs in fact share the same structure. To see how this is the case, we can pull vertex d below vertices a and b. Doing so, we obtain Figure 19. Then, flipping the vertices e and c, we get 20. Lastly, when we rotate 20, we get back to 17.

Definition 3.1. More formally, we say that graphs G and H are *isomorphic* if there is a 1-1 correspondence, denoted by ϕ, between the vertex sets of both graphs and an edge uv exists in $E(G)$ if and only if $\phi(u)\phi(v)$ exists in $\phi(E(G))$.

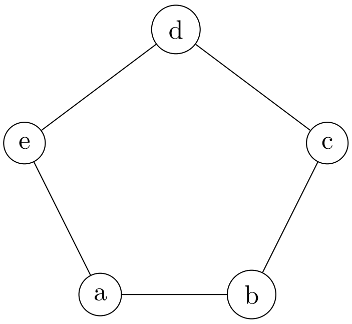

Figure 17. Example 1

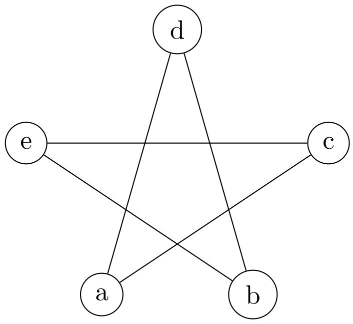

Figure 18. Example 2

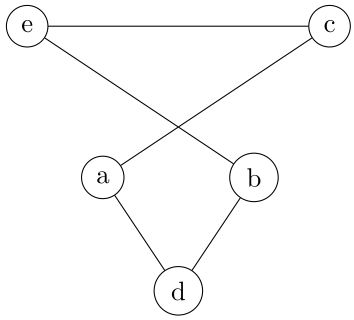

Figure 19. Example 3

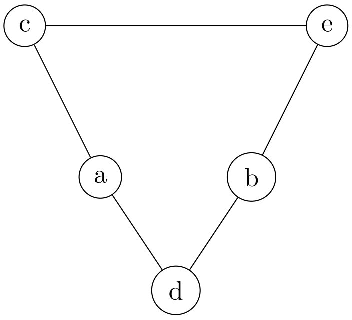

Figure 20. Example 4

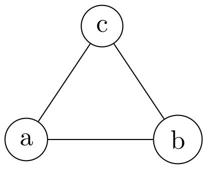

Figure 21. Example 5

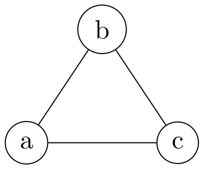

Figure 22. Example 6

3.2 Automorphisms

It is not always immediately apparent when two graphs are isomorphic. In fact, it can be quite difficult to determine if two graphs are isomorphic. However, the easiest isomorphism that always exists is the isomorphism from a graph to itself. This is called an automorphism.

Definition 3.2. An *automorphism* is an isomorphism from a graph G to itself.

Example. An automorphism of Figure 21 could map Figure 21 to Figure 22. To check the conditions, the edges ab, ac, and bc exist in both graphs. Additionally, vertices a, b and c also exist in both graphs. There is also a 1-1 correspondence of the vertices. In these graphs, $\phi(a) = a$, $\phi(b) = c$, and $\phi(c) = b$.

Definition 3.3. The *automorphism group* of a graph G is the set of all automorphisms of G under the function composition operation.

Automorphisms act as a bridge between graph theory and group theory, through the concept of automorphism groups.

4 Applications to City Infrastructure

4.1 Terms and Definitions

When designing a city, an urban planner might think about the placement of important buildings such that these buildings have the greatest utility. For example, a hospital must be accessible from all locations in the city. To optimize the placement of the hospital, one must consider the distance of the hospital from each building in the city. The city can be modeled by a graph, where buildings are vertices and roads are edges. Before we can consider the optimal placement of critical buildings, we present some terminology used to find the optimized ”middle” of a graph.

Definition 4.1. The *distance* between two vertices in a graph is the length of the shortest path between them.

Definition 4.2. The *eccentricity* of a vertex is the maximum of the set of distances to every other vertex.

Definition 4.3. The *diameter* of a graph is the maximum eccentricity of a vertex.

Definition 4.4. The *radius* of a graph is the minimum eccentricity of a vertex.

Definition 4.5. The *center* of a graph is the set of vertices whose eccentricity is the same as the radius.

Example. We use Figure 23 to list some examples of the above terms.

- The eccentricity of vertex a is 3, b is 3, c is 2, d is 2, e is 2, and f is 3.
- The radius of the graph is 2.
- The center is the set $\{c, d, e\}$.

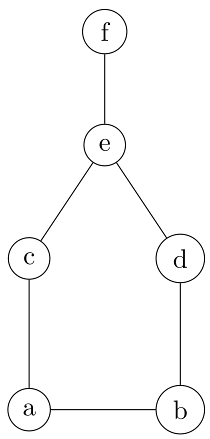

Figure 23. Terms

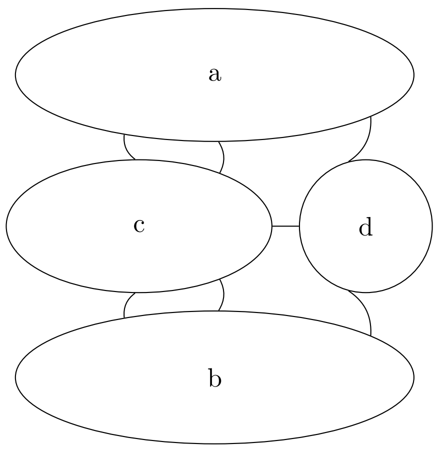

Figure 24. Graph of Koningsberg

By using this method, a hospital can be placed in the center of the graph that represents the city. This guarantees its optimized location.

4.2 Eulerian Graphs

To introduce this idea, let us first present an age old question:

Question 4.6. *The city of Koningsberg is consists of 4 separate lands, with 7 bridges between (see Fig. 24). If a resident of the city would like to take a walk between the four lands and cross every bridge exactly once, is there a way for them to do it?*

We model this situation using a graph. Eventually it was proven that no matter how hard someone tries, they cannot walk in such a way. This famous problem came to be known as the *Euler-Koningsberg* problem once it was brought to Euler's attention.

Definition 4.7. A graph is *Eulerian* only if and only if one can traverse the graph such that each edge is traveled across exactly once.

A city which can be represented by such an Eulerian graph will be able to plan transit routes and delivery routes among other things, much more easily. However, this begs the questions, how do we know a graph is Eulerian? Before planning a delivery route, it is important to know this about a city.

Theorem 4.8. *A graph is Eulerian if and only if it has even degree vertices.*

Proof. For a graph to be Eulerian, we see that each vertex must have a way to enter and to exit. This must mean that a vertex has a certain number of enter-exit pairs and is therefore of even degree. To prove that a graph that is Eulerian must have even degree, we start with induction. Our base case is a multigraph of order 2 and size 2, forming a loop. This graph is still Eulerian. We then assume a multigraph with n order and m size is Eulerian. Now we prove for $n + 1$ vertices. If we are to add a vertex on the graph on an edge, the graph is still Eulerian, because the vertex can be treated as part of an edge. All vertices have even degree. Now we prove for a graph of size $m + 2$. If we create two new edges to and from two of the same vertices, the degree remains even and the graph is still Eulerian. □

This is the reason that a citizen of Koningsberg cannot take a walk that crosses every bridge exactly once. All of the vertices in the graph of Koningsberg are of odd degree! Similarly, a mailman can analyze their delivery route by determining if a graph of their route, where edges are streets and vertices are intersections, is Eulerian. However, a mailman is not constrained by the fact that an Euerian graph must contain a circuit through all edges of the graph. Rather, the only requirement is a trail through all edges in the graph. Therefore, two of the vertices in the route may have odd degree. In particular, these would be the starting and ending points of the route. To make the graph Eulerian, these two vertices would have to have an edge between them.

5 Acknowledgements

We would like to thank our mentor Rachana Madhukara for guiding us in our journey of learning about graph theory and writing this paper. We would also like to thank the PRIMES Circle Coordinator, Dr. Peter Haine, for organizing this program.

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