Sharp Entropy Bounds for Plane Curves and Dynamics of the Curve Shortening Flow

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Julius Baldauf
Mentor: Ao Sun
Project suggested by: Professor W. Minicozzi

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Abstract

We prove that a closed immersed plane curve with total curvature $2\pi m$ has entropy at least m times the entropy of the embedded circle, as long as it generates a type I singularity under the curve shortening flow. Furthermore, we show that there exist closed immersed plane curves of total curvature $2\pi m$ whose entropy is less than m times the entropy of the embedded circle. Our method allows us to calculate the entropy index of all Abresch-Langer curves. Using this, we extend Colding-Minicozzi's notion of a generic MCF to closed immersed plane curves by constructing a piecewise flow whose only singularities are embedded circles and type II singularities.

1 Introduction

We say a family of smoothly immersed hypersufaces $x : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ moves by the *mean curvature flow* (MCF) if it satisfies the equation

$$\left(\frac{\partial x}{\partial t}\right)^{\perp} = -H\mathbf{n}. \tag{1.1}$$

Here H is the mean curvature of the hypersurface $x(\Sigma, t)$, $\mathbf{n} : \Sigma \times [0, T) \to S^n$ is the normal vector field, and $\perp$ denotes projection onto the normal bundle of the immersion. When $n = 1$, this flow is also called *curve shortening flow* (CSF).

Huisken [17] conjectured that for mean curvature flow from generic initial embedded hypersurfaces, all singularities are spheres or cylinders. In a recent fundamental paper [10], Colding and Minicozzi made an important step towards establishing Huisken's genericity conjecture. In that paper, they define the *entropy* of an immersed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ to be

$$\lambda(\Sigma) = \sup_{x_0, t_0} \ (4\pi t_0)^{-n/2} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu, \tag{1.2}$$

where the supremum is taken over all translations $x_0 \in \mathbb{R}^{n+1}$ and rescalings $t_0 > 0$ of Σ. Entropy is a Lyapunov functional for MCF and is constant along a flow precisely for *shrinkers*, the contracting self-similar solutions. Entropy is used in the following important way: if Σ is a shrinker arising as

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the limit of a blowup sequence of a singular point of the MCF starting at the initial hypersurface Σ' then $\lambda(\Sigma) \leq \lambda(\Sigma')$. Entropy thus provides a vital tool for ruling out certain singularities for a MCF.

A singularity of a MCF defined up to time T is of *type I* if the curvature blows up no faster than $(T-t)^{-1/2}$; otherwise, the singularity is of *type II*. Type I singularities are important in the singularity analysis of MCF because Huisken [15] shows that a rescaling of a type I singularity is a smooth shrinker. The first main theorem of this paper gives entropy lower bounds for plane curves in terms of a topological quantity, the *turning number*, which is the total curvature divided by 2π. We denote by $\Gamma_m \subset \mathbb{R}^2$ the *m-covered circle* with radius $\sqrt{2}$, i.e. Γ_m is the map $S^1 \to \mathbb{R}^2$ whose degree is m. Straightforward calculations shows that Γ_m is a shrinker for all $m \geq 1$ and that

$$\text{(1.3)} \qquad \lambda(\Gamma_m) = m \cdot \lambda(\Gamma_1) = m\sqrt{2\pi/e}.$$

In other words, the entropy of the m-covered circle is m times the entropy of the embedded circle of radius $\sqrt{2}$.

Theorem A. *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve with turning number m.*

    A.1 *If the CSF starting at Γ generates a type I singularity, then $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.*

    A.2 *There exist closed immersed curves $\Gamma' \subset \mathbb{R}^2$ with turning number m, but $\lambda(\Gamma') < \lambda(\Gamma_m)$. Any such curve generates only type II singularities under the CSF.*

An important special case of Theorem A.1 is when $\Gamma \subset \mathbb{R}^2$ is a closed shrinker of turning number m. Closed shrinkers for the CSF are called *Abresch-Langer curves* [1]. For any pair of relatively prime integers $m, n \in \mathbb{N}$ satisfying

$$\text{(1.4)} \qquad \frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2},$$

there exists a closed shrinker $\Gamma_{m,n}$ with turning number m and whose curvature has $2n$ critical points. Moreover, every closed shrinker is of this form (other than the round circles). Theorem A.1 implies that

$$\text{(1.5)} \qquad \lambda(\Gamma_{m,n}) \geq \lambda(\Gamma_p)$$

for every Abresch-Langer curve $\Gamma_{m,n}$ and every p-covered circle Γ_p with $m \geq p$. The normalization constant $(4\pi)^{-n/2}$ in the definition of entropy (1.2) is chosen so that hyperplanes have entropy 1. It follows that

$$\text{(1.6)} \qquad \lambda(\Sigma) = \lambda(\Sigma \times \mathbb{R})$$

for all hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$. Using this fact, Theorem A.1 immediately implies entropy bounds on higher dimensional shrinkers $\Sigma \subset \mathbb{R}^{n+1}$ of the form $\Sigma = \Gamma \times \mathbb{R}^{n-1}$, where Γ is a closed plane shrinker.

The key tool for proving Theorem A.2 is the notion of *entropy instability*. A shrinker $\Sigma \subset \mathbb{R}^{n+1}$ is *entropy unstable* if there exists a variation $(\Sigma_\epsilon)_{\epsilon \in (-\delta,\delta)}$ of Σ which decreases entropy. Entropy stability of embedded shrinkers was first introduced in [10]. Our method to construct closed curves with turning number m but entropy less than that of the m-covered circle Γ_m is to show that Γ_m is entropy unstable for all $m \geq 2$. We not only show that multiply-covered circles are entropy unstable, but also that *all* closed shrinkers $\Gamma \subset \mathbb{R}^2$ other than the embedded circle are entropy

unstable. Furthermore, we calculate their *entropy index*, which is the number of linearly independent variations which reduce the entropy of the shrinker (see Liu [20]).

Colding and Minicozzi use entropy stability to define a *piecewise MCF* starting at a closed hypersurface $\Sigma \subset \mathbb{R}^n$ as a finite collection of MCFs Σ_t^i on time intervals $[t_i, t_{i+1}]$ so that each $\Sigma_{t_{i+1}}^{i+1}$ is a graph over $\Sigma_{t_{i+1}}^i$ of a function u_{i+1} and

$$(1.7) \qquad \qquad \lambda(\Sigma_{t_{i+1}}^{i+1}) \leq \lambda(\Sigma_{t_{i+1}}^i).$$

Piecewise MCF provides an ad hoc notion of generic MCF that is important in many applications: it provides a method of continuing a flow through unstable singularities. If a MCF reaches an entropy unstable singularity, the flow is slightly perturbed to decrease entropy so that the singularity can never reoccur along the flow. Colding and Minicozzi construct a piecewise MCF for closed embedded surfaces in $\mathbb{R}^3$ which becomes extinct in a round point [10]. We obtain a similar result by constructing a piecewise CSF for closed immersed curves in $\mathbb{R}^2$.

Theorem B. *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve. Then there exists a piecewise CSF starting at Γ and defined up to time T at which the flow either becomes extinct in an embedded circle, or has type II singularities. Moreover, if Γ has turning number greater than 1, the latter case holds.*

It is worth noting that the break-points of the piecewise CSF constructed in Theorem B can be made arbitrarily small in the C^∞ norm. Furthermore, a result of Gage-Hamilton [12] and Grayson [13] states that the CSF starting at a closed embedded plane curve becomes convex and eventually becomes extinct in a "round point." It follows that the piecewise CSF of Theorem B starting at a closed embedded plane curve reduces to the usual CSF whose only singularity is an embedded circle.

The study of shrinkers is central in the analysis of singularity formation of the MCF for the following reason: every limit of rescalings of a MCF around a fixed point in spacetime is modeled on a possibly singular shrinker [15],[16]. Shrinkers can equivalently be defined as immersed hypersurfaces $\Sigma \subset \mathbb{R}^n$ satisfying the equation

$$(1.8) \qquad \qquad H = \frac{1}{2}\langle x, \mathbf{n} \rangle.$$

Straightfoward calculations show that hyperplanes passing through the origin, round spheres of radius $\sqrt{2n}$ centered at the origin, and cylinders of the form $S^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ are all shrinkers. Much work has concerned the construction and classification of shrinkers [2],[6],[18], and for the case $n = 1$, a complete classification is known [1].

In [22], Stone computes the entropy of embedded spheres (and thus also of shrinking cylinders), implying that

$$(1.9) \qquad \qquad 2 > \lambda(S^1) > \lambda(S^2) > \cdots \to \sqrt{2}.$$

In [5], Brakke proves that hyperplanes have least entropy among all shrinkers, and that there is a gap between the entropy of a plane and the entropy of the next lowest shrinker. Colding-Ilmanen-Minicozzi-White [8] prove that the sphere is the shrinker with second lowest entropy, and that a sphere has least entropy of among all closed embedded shrinkers. This result has been generalized by Bernstein-Wang [4], Zhu [24], and Ketover-Zhou [19] to all closed embedded hypersurfaces, not only shrinkers. Very recently, Hershkovits and White [14] proved a lower bound and rigidity theorem for shrinkers: a closed embedded shrinker $\Sigma \in \mathbb{R}^{n+1}$ with non-trivial k-th homology has

entropy greater than or equal to the entropy of S^k, and equality holds if and only if $\Sigma = S^k \times \mathbb{R}^{n-k}$. Our results can be viewed as an extension of these results to the case of closed *immersed* curves.

Stability of Abresch-Langer curves has been studied in a different context by Epstein-Weinstein [11]. Our results show that the Jacobi operator of an Abresch-Langer curve has the same number of negative eigenvalues as the linear stability operator studied by Epstein-Weinstein. Their analysis does not incorporate the action of the group generated by rigid motions and dilations. In particular, for a curve to be in Epstein-Weinstein's stable manifold, then under the rescaled flow it has to limit into the given shrinker. In other words, for a curve to be in their stable manifold, it is not enough that it limit into a rotation, translation, or dilation of the given shrinker. However, in a *generic MCF* [10], a curve which limits into a rotation, translation, or dilation of a given shrinker, is considered as part of the stable manifold.

Outline of the Proofs.   We now outline the proof of Theorem A.1. The main technical challenge is showing that the entropy functional is continuous under small perturbations near a closed shrinker Γ. Note that the entropy functional not continuous in general: a sequence of rescalings at a point on a sphere converges to a hyperplane in the limit, however, a hyperplane has entropy 1, while a sphere has entropy at least $\sqrt{2}$. Our method for showing that the entropy functional is continuous near a closed shrinker $\Gamma \subset \mathbb{R}^2$ consists of two steps. Firstly, Proposition 2.1 is a general result giving a sufficient condition for the F-functional of a closed immersed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ to be continuous at Σ, namely, when (x_0, t_0) is restricted to a compact subset $K \subset \mathbb{R}^{n+1} \times (0, \infty)$ and variations are restricted to a subset $\mathcal{B} \subset C^{2,\alpha}(\Sigma)$ of variations with a uniform $C^{2,\alpha}$ bound.

The second step is to provide a quantitative version of [10, Lemma 7.7] to show that we may in fact apply Proposition 2.1 to closed shrinkers $\Gamma \subset \mathbb{R}^2$. To that end, Theorem 2.3 shows that the entropy of the perturbed curve $\Gamma + f\mathbf{n}$ with $f \in \mathcal{B}$ is achieved in $B_R(0) \times [T_0, T_1]$ for some $R > 0$ and some $0 < T_1 < T_2 < \infty$. The main technical challenge is to show that $T_1 > 0$. The key result we use to conclude the proof is due to Au [3]. Au shows that if $\Gamma_{m,n}$ is an Abresch-Langer curve, then when $\epsilon > 0$ is small enough, the perturbed curve $\Gamma_{m,n} + \epsilon\mathbf{n}$ converges to the m-covered circle Γ_m under the rescaled CSF. Using Au's result and the monotonicity property of entropy we conclude the proof.

Next, we give an outline of the proof of Theorem A.2. Our method to construct closed curves with turning number m but entropy less than that of the m-covered circle Γ_m is to show that Γ_m is entropy unstable for all $m \geq 2$. An important tool for analyzing the entropy stability of a closed hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is its F-*functional*

$$(1.10) \qquad\qquad F_{x_0, t_0}(\Sigma) = (4\pi t_0)^{-n/2} \int_\Sigma e^{-\frac{|x - x_0|^2}{4t_0}} d\mu.$$

The entropy $\lambda(\Sigma)$ is then the supremum over all translations $x_0 \in \mathbb{R}^{n+1}$ and dilations $t_0 > 0$ of the F-functional. It follows from Huisken's monotonicity formula [15] that shrinkers are precisely the critical points of the $F_{0,1}$-functional, the *Gaussian area*. Therefore, shrinkers are precisely the minimal hypersurfaces of $\mathbb{R}^{n+1}$ equipped with the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij}$. We show that the entropy index of a shrinker is the difference between the Morse index[1] of the shrinker (considered as a minimal surface of $(\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{4}} \delta_{ij})$) and the dimension of the space spanned by the mean curvature function and the component functions of the normal vector field

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[1]Recall that the Morse index of a minimal surface $\Sigma \subset (M, g)$ is defined as the number of negative eigenvalues of the Jacobi operator determined by (M, g) and Σ (see [9, p. 41]).

of the shrinker. Consequently, we reduce the calculation of the entropy index of a shrinker to the calculation of the Morse index. For 1-dimensional shrinkers in $\mathbb{R}^2$, the Jacobi operator reduces to a 1-dimensional Sturm-Liouville operator. Using the well-developed theory of spectra of Sturm-Liouville operators, we are able to determine exactly how many negative eigenvalues the Jacobi operator of a 1-dimensional closed shrinker has, and thus also the entropy index.

Lastly, we outline the proof of Theorem B. As mentioned above, the possible singularities of the CSF starting at Γ are classified into two categories: type I and type II. Huisken [15, Theorem 3.5] shows that any rescaling of a type I singularity is a shrinker. By Abresch and Langer's classification of shrinkers for the CSF [1], any rescaling of a type I singularity must therefore be a Abresch-Langer curve, or a (multiply-covered) circle. Corollary 5.8 states that the only entropy stable closed singularity for CSF is the embedded circle. Let Γ be any closed immersed plane curve. If the singularity is of type I, any rescaling gives an Abresch-Langer curve or a circle, say $\widetilde{\Gamma}_\infty$. If $\widetilde{\Gamma}_\infty$ is not an embedded circle, Lemma 6.1 shows that the flow can be slightly perturbed to a curve Γ' with $\lambda(\Gamma') < \lambda(\widetilde{\Gamma}_\infty)$. Consequently, $\widetilde{\Gamma}_\infty$ can never appear as a singularity for the flow starting from Γ'. The above process is then repeated with Γ' instead of Γ. Since the perturbations can be made with arbitrarily small C^∞-norm, the perturbations preserve turning number. Therefore piecewise CSF preserves turning number, and since there are only finitely many closed shrinkers of a given turning number, the piecewise flow terminates after finitely many perturbations.

Organization of the Paper.   In Section 2 we prove some basic properties of the entropy functional for immersed hypersurfaces and shrinkers. In particular, here we prove the continuity Theorem 2.3 for entropy, which is central to our argument.

In Section 3, we prove Theorem A.1.

In Section 4, we introduce and study some basic properties of index of the shrinkers. The results of this section hold for shrinkers of any dimension.

In Section 5, we compute the precise entropy index of closed immersed shrinkers for the CSF. As a corollary, we obtain Theorem A.2.

In Section 6 we combine our results to prove Theorem B.

Notation.   Throughout, all hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ are assumed to be immersed and orientable, so that there exists a globally defined unit normal vector field $\mathbf{n} : \Sigma \to S^n$. We denote by $x : \Sigma \to \mathbb{R}^{n+1}$ the given immersion and by $\Sigma + f\mathbf{n}$ the normal variation of Σ by a function $f : \Sigma \to \mathbb{R}$. The mean curvature of Σ is denoted by $H : \Sigma \to \mathbb{R}$. It is easy to show (see for example [21]) that MCF is a geometric flow which is invariant under tangential reparameterization, so without ambiguity we may also use the image $(\Sigma_t)_{t\in[0,T)}$ to denote the flow. In the case of curves, we use the notation $(\Gamma_t)_{t\in[0,T)}$. Finally, angle brackets $\langle \cdot , \cdot \rangle$ denote the standard Euclidean inner product.

2   Properties of Entropy

In this section, we will provide conditions under which the entropy functional is continuous at a closed plane shrinker. Note that in general, the entropy functional is not a continuous function on the space of immersed hypersurfaces (with the C^∞ topology, say). The standard example is given by blowing up a sphere at a point, giving a hyperplane in the limit. Each element of the blowup sequence is a sphere, thus having entropy greater than $\sqrt{2}$. However, a hyperplane has entropy 1. The results of this section will be used in the Section 3 to prove Theorem A.1.

The following proposition shows that, when a closed hypersurface is slightly perturbed, the F-functional cannot change too much under the perturbation, as long as (x_0, t_0) are confined to a compact subset of $\mathbb{R}^{n+1} \times (0, \infty)$.

Proposition 2.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed hypersurface and $K \subset \mathbb{R}^{n+1} \times (0, \infty)$ a compact subset. Then there exists an $\epsilon_0 > 0$ and a constant C, both depending only on Σ and K, such that, for all $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \le 1$, all $\epsilon \ge 0$ with $\epsilon < \epsilon_0$, and all $(x_0, t_0) \in K$, the following inequality holds:*

$$(2.2) \qquad\qquad |F_{x_0,t_0}(\Sigma_\epsilon) - F_{x_0,t_0}(\Sigma)| \le C\epsilon,$$

where $\Sigma_\epsilon = \Sigma + \epsilon f \mathbf{n}$.

Proof. Let $x : \Sigma \to \mathbb{R}^{n+1}$ be the given immersion and let $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \le 1$. In local coordinates around a point $p \in \Sigma$, we can write

$$\widetilde{g}_{ij} = \left\langle \partial_i(x + \epsilon f \mathbf{n}), \, \partial_j(x + \epsilon f \mathbf{n}) \right\rangle$$
$$= g_{ij} + 2\epsilon f h_{ij} + \epsilon^2 f^2 h_{ik} h^k_j + \epsilon^2 (\partial_i f)(\partial_j f),$$

where g_{ij} is the metric induced on Σ as a hypersurface in $\mathbb{R}^{n+1}$. Since Σ is closed, it follows that there exists an $\epsilon_0 > 0$, such that, for any $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \le 1$ and any $\epsilon \in \mathbb{R}$ satisfying $|\epsilon| < \epsilon_0$, the tensor $\widetilde{g}$ defines a metric on Σ_ϵ and the volume measure $\sqrt{\widetilde{g}}$ and the derivative $\partial \widetilde{g}/\partial \epsilon$ are uniformly bounded on compact sets. Define

$$G_{x_0,t_0}(\beta, x) = e^{-\frac{|x - x_0 + \beta f \mathbf{n}|^2}{4t_0}} \sqrt{\widetilde{g}}.$$

Then Jacobi's formula gives

$$\frac{\partial G_{x_0,t_0}(\beta, x)}{\partial \beta} = \frac{1}{2}\left(\mathrm{Tr}\left(\widetilde{g}^{-1} \frac{\partial \widetilde{g}}{\partial \beta} \right) - \frac{f \langle x - x_0, \mathbf{n} \rangle + \beta f^2}{t_0} \right) e^{-\frac{|x - x_0 + \beta f \mathbf{n}|^2}{4t_0}} \sqrt{\widetilde{g}},$$

which is uniformly bounded on K by a constant depending only on ϵ_0 and K. In particular, the

constant is independent of the choice of $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$. Therefore

$$
\begin{aligned}
|F_{x_0,t_0}(\Sigma_\epsilon) - F_{x_0,t_0}(\Sigma)| &= \left| \frac{1}{\sqrt{4\pi t_0}} \int_\Sigma (G_{x_0,t_0}(\epsilon, x) - G_{x_0,t_0}(0, x)) dx \right| \\
&\leq \frac{1}{\sqrt{4\pi t_0}} \int_\Sigma |G_{x_0,t_0}(\epsilon, x) - G_{x_0,t_0}(0, x)| \, dx \\
&= \frac{\epsilon}{\sqrt{4\pi t_0}} \int_\Sigma \left| \int_0^1 \frac{\partial G_{x_0,t_0}(\epsilon u, x)}{\partial \beta} du \right| dx \\
&\leq \frac{\epsilon}{\sqrt{4\pi t_0}} \int_\Sigma \int_0^1 \left| \frac{\partial G_{x_0,t_0}(\epsilon u, x)}{\partial \beta} \right| du \, dx \\
&\leq \epsilon C
\end{aligned}
$$

as desired. $\qquad\square$

The main result of this section, below, shows there exists a compact subset of $\mathbb{R}^2 \times (0, \infty)$ such that the entropy of any curve obtained by perturbing a given closed plane shrinker is attained in this compact set. As a consequence, we may apply the previous proposition to conclude that the entropy functional is continuous under small perturbations of a closed plane shrinker. Theorem 2.3 may be viewed as a quantitative version of [10, Lemma 7.7].

Theorem 2.3. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker. Then there exists an $\epsilon_0 > 0$ and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$, both depending only on Γ, such that, for any variation $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$, the following holds:*

$$
\lambda(\Gamma_\epsilon) = \sup_{(x_0, t_0) \in K} F_{x_0, t_0}(\Gamma_\epsilon),
$$

where $\Gamma_\epsilon = \Gamma + \epsilon f \mathbf{n}$.

Proof. Throughout, if $f \in C^{2,\alpha}(\Gamma)$, then we will denote by Γ_ϵ^f the perturbed curve $\Gamma_\epsilon^f = \Gamma + \epsilon f \mathbf{n}$. As stated in the proof of the previous proposition, there exists an $\epsilon_0 > 0$ such that Γ_ϵ^f is a closed immersed curve for any $f \in C^{2,\alpha}(\Gamma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$. Let $\mathcal{B} \subset C^{2,\alpha}(\Gamma) \times (0, \infty)$ consist of those Γ_ϵ^f with $f \in C^{2,\alpha}(\Sigma)$ such that $\|f\|_{C^{2,\alpha}} \leq 1$ and $\epsilon \in \mathbb{R}$ such that $|\epsilon| < \epsilon_0$.

Step (1): Bounds on $|x_0|$. By the compactness of Γ and the uniform $C^{2,\alpha}$ bound the the variation functions, there exists $R > 0$ (independent of f and ϵ) such that Γ_ϵ^f is contained in the closed ball $B_R(0) \subset \mathbb{R}^2$ of radius R centered at the origin, which is compact. Colding and Minicozzi [10] show that for any closed hypersurface $M \subset \mathbb{R}^{n+1}$ and each fixed $t_0 > 0$, $\sup_{x_0 \in \mathbb{R}^{n+1}} F_{x_0, t_0}(M)$ is achieved inside the convex hull of M, which is compact, since M is closed. Therefore, for each fixed t_0 and all $\Gamma_\epsilon^f \in \mathcal{B}$, $\sup_{x_0 \in \mathbb{R}^2} F_{x_0, t_0}(\Gamma_\epsilon^f)$ is attained in $B_R(0) \subset \mathbb{R}^2$.

Step (2): Upper bound on t_0. Let $T_1 = (4\pi)^{-n/2} \mathrm{Vol}(B_R)$. Then for any $t_0 > T_1$, any $x_0 \in B_R(0)$, and any hypersurface $\Gamma_\epsilon^f \in \mathcal{B}$, we have

$$
\begin{aligned}
F_{x_0, t_0}(\Gamma_\epsilon^f) &\leq (4\pi t_0)^{-n/2} \mathrm{Length}(\Gamma_\epsilon^f) \\
&\leq (4\pi t_0)^{-n/2} \mathrm{Vol}(B_R) \\
&< (4\pi T_1)^{-n/2} \mathrm{Vol}(B_R) \\
&= 1.
\end{aligned}
$$

However, we know that $\lambda(\Gamma_\epsilon^f) > 1$. This can be seen as follows: a closed embedded curve has entropy at least that of the round circle, while a closed curve with self-intersection has entropy at least 2. We conclude that

$$\sup_{(x_0, t_0) \in B_R(0) \times (0, \infty)} F_{x_0, t_0}(\Gamma_\epsilon^f) = \sup_{(x_0, t_0) \in B_R(0) \times (0, T_1]} F_{x_0, t_0}(\Gamma_\epsilon^f),$$

for all $\Gamma_\epsilon^f \in \mathcal{B}$.

Step (3): Lower bound on t_0. We observe that for ϵ_0 small enough, Γ_ϵ^f has a uniform curvature bound. This follows from the expression of curvature in terms of the first and second derivatives of the immersion. Therefore, we may assume that $|k| \leq k_0$ for all $\epsilon \leq \epsilon_0$. Let $R_0 > 0$ be a constant whose value will be determined. Now for any fixed x_0 and any $t_0 \leq R_0^{-3}$, we have

$$F_{x_0, t_0}(\Gamma_\epsilon^f) = (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \geq R_0^{-1}\} \cap \Gamma_\epsilon^f} e^{-\frac{|x-x_0|^2}{4t_0}} ds$$

$$+ (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \leq R_0^{-1}\} \cap \Gamma_\epsilon^f} e^{-\frac{|x-x_0|^2}{4t_0}} ds.$$

The first term on the right hand side is bounded by

$$(4\pi t_0)^{-1/2} e^{-\frac{R_0^{-2}}{4t_0}} \text{Length}(\Gamma_\epsilon^f) \leq C R_0^{-1/2}$$

for a universal constant C.

Now we estimate the second term on the right hand side. By change of variables, this term equals

$$\int_{\{|x-x_0| \leq R_0^{-1} t_0^{-1}\} \cap (t_0^{-1} \Gamma_\epsilon^f)} e^{-\frac{|x-x_0|^2}{4}} ds,$$

where $t_0^{-1} \Gamma_\epsilon^f$ is the curve Γ_ϵ^f rescaled by t_0^{-1} with center at x_0. Suppose $\{|x - x_0| \leq R_0^{-1} t_0^{-1}\} \cap (t_0^{-1} \Gamma_\epsilon^f)$ consists of the connected components $\gamma_1, \ldots, \gamma_m$. Then each γ_i is a curve with curvature uniformly bounded by $k_0 t_0$. In particular, for a given $\delta > 0$, when R_0 is large enough depending on k_0, the curvature of γ_i is small enough and therefore

$$\int_{\gamma_i} e^{-\frac{|x-x_0|^2}{4}} ds \leq 1 + \delta.$$

So we only need to bound the number of such curves to bound the second term.

Let us consider the immersed tubular neighborhood $N(\Gamma, \alpha)$ of Γ of radius α, which is an open subset of $\mathbb{R}^2$ with multiplicity. Here α only depends on Γ and can be chosen small enough such that the multiplicity of the tubular neighborhood is at most the multiplicity of the curve Γ. Let us choose ϵ_0 small such that the $\epsilon_0 < \alpha/2$, and choose R_0 large such that $R_0^{-1} < \alpha/2$, then if the ball $\{|x - x_0| \leq R_0^{-1}\}$ intersects Γ_ϵ^f nontrivially, it completely lies in $N(\Gamma, \alpha)$. Moreover, each γ_i counts the multiplicity of the tubular neighborhood $N(\Gamma, \alpha)$ once. So the number m of γ_i's is no more than the multiplicity of $N(\Gamma, \alpha)$, hence no more than the multiplicity of Γ.

We conclude that

$$F_{x_0, t_0}(\Gamma_\epsilon^f) \leq C R_0^{-1/2} + m(1 + \delta) = m + \delta m + C R_0^{-1/2}.$$

Since Γ is a shrinker, a result of Colding-Minicozzi [10, Lemma 7.7] gives that its entropy is achieved at $F_{0,1}$. Thus the entropy of Γ is greater than the multiplicity of Γ, i.e. $\lambda(\Gamma) > m + \beta$ for

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some $\beta > 0$. So when ϵ_0 is small, Proposition 2.1 gives that $F_{0,1}(\Gamma_\epsilon^f) > m + \beta/2$. If we pick δ small enough and R_0 large enough, for $t_0 \leq R_0^{-3} =: T_0$ we have

$$F_{x_0,t_0}(\Gamma_\epsilon^f) \leq m + \beta/4 < m + \beta/2 < \lambda(\Gamma).$$

Combining all of the steps above, the proposition follows, with $K := B_R(0) \times [T_0, T_1]$. $\qquad\square$

Corollary 2.4. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker and let $(\Gamma_i)_{i\in\mathbb{N}}$ be a sequence of plane curves which C^∞ converges to Γ. Then*

$$\lim_{i\to\infty} \lambda(\Gamma_i) = \lambda(\Gamma).$$

Proof. It follows from the convergence assumption that there exists an $N' \in \mathbb{N}$, such that, for $i > N'$, Γ_i can be written as a graph

$$\Gamma_i = \Gamma + \epsilon_i f_i \mathbf{n},$$

for some $f_i \in C^\infty(\Gamma)$ normalized to $\|f_i\|_{C^\infty} = 1$ and some $\epsilon_i > 0$ with $\epsilon_i \to 0$ as $i \to \infty$. Applying the previous theorem implies that there exists an $N'' \geq N'$ and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$ such that, for all $i > N''$, the entropy $\lambda(\Gamma_i)$ is achieved at some $(x_i, t_i) \in K$. Then by Proposition 2.1, there exists an $N \geq N''$ and a constant C depending only on K such that, for any $(x_0, t_0) \in K$ and any $i > N$, we have

$$F_{x_0,t_0}(\Gamma_i) - C\epsilon_i \leq F_{x_0,t_0}(\Gamma) \leq F_{x_0,t_0}(\Gamma_i) + C\epsilon_i.$$

Consequently,

$$\sup_K F_{x_0,t_0}(\Gamma_i) - C\epsilon_i \leq \sup_K F_{x_0,t_0}(\Gamma) \leq \sup_K F_{x_0,t_0}(\Gamma_i) + C\epsilon_i.$$

Since the entropy $\lambda(\Gamma_i)$ is achieved in K for all $i > N$, it follows immediately that

$$\lambda(\Gamma_i) - C\epsilon_i \leq \lambda(\Gamma) \leq \lambda(\Gamma_i) + C\epsilon_i.$$

Now taking the limit $i \to \infty$ gives the desired result. $\qquad\square$

The following corollary will be central to our construction of a generic CSF. The previous corollary shows that the entropy is continuous *at* a closed plane shrinker. However, for the construction of the piecewise CSF in Theorem B, we also need the entropy to be continuous *near* a closed plane shrinker. The following corollary ensures that the latter condition is satisfied.

Corollary 2.5. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker, $f \in C^{2,\alpha}(\Gamma)$ a variation, and $(g_i)_{i\in\mathbb{N}} \subset C^\infty(\Gamma)$ a sequence of functions which C^∞-converges to 0. Then there exists an $\epsilon_0 > 0$ such that, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$, we have*

$$\lim_{i\to\infty} \lambda(\Gamma_\epsilon + g_i \mathbf{n}) = \lambda(\Gamma_\epsilon),$$

where $\Gamma_\epsilon = \Gamma + \epsilon f \mathbf{n}$.

Proof. By the C^∞ convergence assumption, there exists $N > 0$ such that $\|g_i\|_{C^{2,\alpha}} \leq 1/2$ for all $i > N$. Since Γ is compact and $f \in C^{2,\alpha}(\Gamma)$, there exists $\epsilon_1 > 0$ such that $\|\epsilon f\|_{C^{2,\alpha}} \leq 1/2$ for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_1$. Consequently, $\|\epsilon f + g_i\|_{C^{2,\alpha}} \leq 1$ for all $i > N$ and all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_1$. By Theorem 2.3, there exists an $\epsilon_2 > 0$ with $\epsilon_2 \leq \epsilon_1$, an $N' \geq N$, and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$ such that, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_2$ and all $i > N'$, the entropy of the curves Γ_ϵ

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and $\Gamma_\epsilon + g_i\mathbf{n}$ is attained in K. By Proposition 2.1, for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$ and all $(x_0, t_0) \in K$, we have

$$\lim_{i \to \infty} F_{x_0, t_0}(\Gamma_\epsilon + g_i\mathbf{n}) = F_{x_0, t_0}(\Gamma_\epsilon).$$

Since the curves Γ_ϵ and $\Gamma_\epsilon + g_i\mathbf{n}$ attain their entropies in K for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$ and all $i > N'$, arguing as in the proof of the previous corollary concludes the proof. $\qquad\square$

3   Entropy and Turning Number

In this section we will prove Theorem A.1, which gives lower bounds on the entropy of closed plane curves which generate type I singularities under the CSF. We will first prove a particular case, in which we only consider the closed plane shrinkers themselves.

Theorem 3.1. *Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker with turning number m. Then $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.*

Proof. We assume $\Gamma \neq \Gamma_m$, since otherwise the result holds trivially. Let $x : S^1 \to \mathbb{R}^2$ be the given immersion of Γ. There are two cases to consider.

Case (1): Suppose that the degree of the map $x : S^1 \to x(S^1)$ is 1 (in other words, Γ is not multiply-covered). Define the constant normal variation function $f = 1$ along Γ. By Corollary 2.4, if $\Gamma_\epsilon = \Gamma + \epsilon\mathbf{n}$, then

$$\lim_{\epsilon \to 0} \lambda(\Gamma_\epsilon) = \lambda(\Gamma).$$

In particular, there exists a constant $C > 0$ such that

$$\lambda(\Gamma) - C\epsilon \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon,$$

for all $\epsilon > 0$ small enough. Au [3] shows that the rescaled CSF starting at Γ_ϵ converges to the the m-covered circle. By the monotonicity property of entropy, we therefore have

$$\lambda(\Gamma_m) \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon.$$

Taking the limit $\epsilon \to 0$ shows that $\lambda(\Gamma) \geq \lambda(\Gamma_m)$.

Case (2): Suppose that the degree of the map $x : S^1 \to x(S^1)$ is greater than 1 (in other words, Γ is a multiply-covered Abresch-Langer curve). Then Γ is a k-covered Abresch-Langer curve (with $k \geq 2$) Γ', where Γ' has turning number p satisfying $kp = m$. By the previous case and (1.3),

$$\lambda(\Gamma) = k \cdot \lambda(\Gamma') \geq k \cdot \lambda(\Gamma_p) = kp \cdot \lambda(\Gamma_1) = m \cdot \lambda(\Gamma_1) = \lambda(\Gamma_m).$$

$\qquad\square$

Lemma 3.2. *The turning number of a closed immersed curve $\Gamma \subset \mathbb{R}^2$ is preserved under the CSF and rescaled CSF.*

Proof. CSF and rescaled CSF preserve immersedness and are therefore regular homotopies. Regular homotopy classes of immersions $S^1 \to \mathbb{R}^2$ are classified by their turning number, by the Whitney-Graustein theorem [23, Theorem 1]. In other words, two closed immersed plane curves are regularly homotopic if and only if they have the same turning number. $\qquad\square$

Proof of Theorem A.1. We assume that Γ is not a shrinker, since otherwise the theorem follows immediately from Theorem 3.1. Let $(\Gamma_t)_{t\in[0,T)}$ be the CSF starting at Γ. Since all type I singularities are closed shrinkers, they are compact, so there exists a unique singular point $x_0 \in \mathbb{R}^2$. Without loss of generality, we may assume x_0 is the origin. Let $(\widetilde{\Gamma}_\tau)_{\tau\in[\log T^{-1/2},\infty)}$ be a rescaled CSF [15, Section 2] around $0 \in \mathbb{R}^2$ starting at Γ. By [15, Theorem 3.5], for any sequence $\tau_i \to \infty$, there exists a subsequence also denoted τ_i such that the rescaled curves $\widetilde{\Gamma}_{\tau_i}$ will smoothly converge to a shrinker $\widetilde{\Gamma}_\infty$ as $\tau_i \to \infty$. By the classification of 1-dimensional shrinkers [1], $\widetilde{\Gamma}_\infty$ must be a multiply-covered circle or a multiply-covered Abresch-Langer curve.

Let n be the turning number of $\widetilde{\Gamma}_\infty$. By the lemma, the turning number of $\widetilde{\Gamma}_\tau$ is m for all $\tau \in [\log T^{-1/2}, \infty)$. By the C^∞-convergence $\widetilde{\Gamma}_{\tau_i} \to \widetilde{\Gamma}_\infty$, the curvature of $\widetilde{\Gamma}_{\tau_i}$ smoothly converges to the curvature of $\widetilde{\Gamma}_\infty$. Therefore the total curvatures satisfy

$$n = \lim_{i\to\infty} \frac{1}{2\pi} \int_{\widetilde{\Gamma}_{\tau_i}} \widetilde{k}_i = m.$$

This shows that $\widetilde{\Gamma}_\infty$ is a shrinker of turning number m. By the monotonicity of entropy,

$$\lambda(\widetilde{\Gamma}_\tau) \leq \lambda(\Gamma)$$

for all $\tau \in [\log T^{-1/2}, \infty)$. By the C^∞-convergence $\widetilde{\Gamma}_{\tau_i} \to \widetilde{\Gamma}_\infty$, Corollary 2.4 implies that

$$\lim_{i\to\infty} \lambda(\widetilde{\Gamma}_{\tau_i}) = \lambda(\widetilde{\Gamma}_\infty),$$

and since this holds for any blowup sequence, it follows thus that

$$\lambda(\Gamma) \geq \lambda(\widetilde{\Gamma}_\infty).$$

Since we assume that Γ is not a shrinker and since entropy is constant along the CSF only for shrinkers, the inequality is in fact strict. The theorem now follows from Theorem 3.1. $\qquad\square$

4 Entropy Index and Morse Index

Recall from the introduction that the shrinkers $\Sigma \subset \mathbb{R}^{n+1}$ are precisely the minimal surfaces of $\mathbb{R}^{n+1}$ equipped with the conformally changed metric $g_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij}$. In this section, we will provide an important connection between the entropy stability of a shrinker and the stability of the shrinker considered as a minimal surface in $(\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{4}} \delta_{ij})$. Namely, Theorem 4.7, shows that a closed shrinker $\Sigma \subset \mathbb{R}^{n+1}$ is entropy unstable if and only if the Morse index of Σ is greater than $n + 2$. Consequently, to prove Theorem A.2 we will only need to show that the m-covered circle has Morse index greater than 3. The results of this section apply to shrinkers $\Sigma^n \subset \mathbb{R}^{n+1}$ of any dimension.

Definition 4.1 (Entropy stability). Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a closed shrinker.

(i) Let Σ_ϵ be a variation of Σ in the direction $f \in C^{2,\alpha}(\Sigma)$. The variation Σ_ϵ is *entropy stable* if there exist some variations x_ϵ of 0 and t_ϵ of 1 such that

$$\frac{\partial^2}{\partial \epsilon^2}\bigg|_{\epsilon=0} F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon) \geq 0;$$

otherwise, the variation is *entropy unstable*.

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(ii) The shrinker Σ is *entropy stable* if every variation of Σ is entropy stable. Otherwise, Σ is *entropy unstable*.

(iii) The *entropy index* of Σ is the number of linearly independent unstable variations.

When there is no possibility for confusion, we simply say a variation or a shrinker is stable (unstable), instead of entropy stable (entropy unstable). In [10, Theorem 0.15], Colding-Minicozzi show that for an entropy unstable shrinker, there exists a variation which will decrease entropy, thus providing the link between entropy and F-functional. Their theorem is stated for embedded shrinkers, but their proof also works for closed immersed shrinkers.

Theorem 4.2 ([10])**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker whose entropy index is nonzero. Then Σ is entropy unstable, i.e. there exists a variation $f \in C^\infty(\Sigma)$ and an $\epsilon_0 > 0$ such that $\lambda(\Sigma_\epsilon) < \lambda(\Sigma)$ for all $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$ and $\epsilon \neq 0$, where $\Sigma_\epsilon = \Sigma + \epsilon f \mathbf{n}$.*

We will use two lemmas in the proof of Theorem 4.7:

Lemma 4.3. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker and let $f \in C^{2,\alpha}(\Sigma)$ be contained in the subspace of variations spanned by H and the components $\mathbf{n}_1, \ldots, \mathbf{n}_{n+1}$ of the normal vector field $\mathbf{n} : \Sigma \to \mathbb{R}$. Then f is an entropy stable variation.*

Proof. We refer the reader to [10, Section 4] for properties of the second variation of the F-functional. Let $f = aH + \langle Y, \mathbf{n} \rangle$ for some $a \in \mathbb{R}$ and some $Y \in \mathbb{R}^{n+1}$. Choose any $y \in \mathbb{R}^{n+1}$ and any $h \in \mathbb{R}$ and set $x_\epsilon = \epsilon y$, $t_\epsilon = 1 + \epsilon h$ and $\Sigma_\epsilon = \Sigma + \epsilon f \mathbf{n}$. Then with $F'' = \partial_{\epsilon\epsilon}\big|_{\epsilon=0}(F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon))$, we have

$$
F'' = \int_\Sigma \left(-fLf + 2hfH - h^2 H^2 + f\langle y, \mathbf{n} \rangle - \frac{1}{2}\langle y, \mathbf{n} \rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu
$$

$$
= \int_\Sigma \left(-a^2 H^2 - \frac{1}{2}\langle Y, \mathbf{n} \rangle^2 + 2ahH^2 - h^2 H^2 + \langle Y\mathbf{n}, \rangle\langle y, \mathbf{n} \rangle - \frac{1}{2}\langle y, \mathbf{n} \rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu
$$

$$
= -\int_\Sigma \left((a-h)^2 H^2 + \frac{1}{2}\left(\langle Y, \mathbf{n} \rangle - \langle y, \mathbf{n} \rangle \right)^2 \right) e^{-\frac{|x|^2}{4}} d\mu.
$$

Thus, if we choose $h = a$ and $y = Y$, then $F'' = 0$, so f is an entropy stable variation. $\qquad\square$

Lemma 4.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed, orientable hypersurface. Then the component functions $\mathbf{n}_i : \Sigma \to \mathbb{R}$ of the unit normal vector field $\mathbf{n}$ are linearly independent.*

Proof. We claim that if the Gauss map $G : \Sigma \to S^n$ is surjective, then the result holds. Suppose that the Gauss map is surjective, and suppose for contradiction that the $\mathbf{n}_i$ are linearly dependent, i.e. there exists $y \in \mathbb{R}^{n+1}$ such that $\langle y, \mathbf{n}(p) \rangle = 0$ for all $p \in \Sigma$. Then since the Gauss map is surjective, y is orthogonal to every vector in $\mathbb{R}^{n+1}$, which is true if and only if y is the zero vector.

Now we show that Gauss map of any closed orientable hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is surjective. Pick any vector $v \in S^n$. We wish to show that there exists some $p \in M$ such that $\mathbf{n}(p) = v$. By the compactness of M, the smooth function $\langle v, x \rangle : M \to \mathbb{R}$ attains a maximum at some point $p \in M$. Let $\{e_i\}$ be a basis of $T_p M$. Then

$$
0 = \nabla_{e_i} \langle v, x(p) \rangle = \langle v, e_i \rangle
$$

for all $i = 1, \ldots, n$. Consequently, v is orthogonal to M at p, so $v = \pm \mathbf{n}(p)$. Since we have chosen p to be the maximum, and since $\langle v, x \rangle$ is not constant (M is closed), we must in fact have $v = \mathbf{n}(p)$, as desired. $\qquad \square$

Corollary 4.5. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker. Then*

$$(4.6) \qquad \dim \operatorname{span}\{H, \mathbf{n}_1, \ldots, \mathbf{n}_{n+1}\} = n + 2,$$

where $\mathbf{n}_i$ is the ith component function of the unit normal vector field $\mathbf{n}$.

Proof. Note that H and $\mathbf{n}_i$ are linearly independent for all i, since $LH = H$ and $L\mathbf{n}_i = \frac{1}{2}\mathbf{n}_i$. The proposition now follows from Lemma 4.4. $\qquad \square$

We are now able to prove the main result of this section, which will be used in the next section to calculate the entropy index of Abresch-Langer curves and m-covered circles.

Theorem 4.7. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed shrinker with Morse index m. Then Σ has entropy index*

$$(4.8) \qquad \Lambda(\Sigma) = m - n - 2.$$

Proof. By standard elliptic theory, the Morse index of a compact minimal surface is finite. In particular, m is finite.

First we will show that the entropy index is at most $m - n - 2$. By the previous proposition, this will follow immediately if we can show that the entropy index is bounded above by the Morse index. Suppose f is an entropy unstable variation of Σ. Then by definition, for any variations x_ϵ of $0 \in \mathbb{R}^{n+1}$ and t_ϵ of 1, we have $(F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} < 0$. In particular, this holds for the trivial variations $x_\epsilon = 0$ and $t_\epsilon = 1$:

$$(F_{0,1}(\Sigma_\epsilon))''|_{\epsilon=0} = -\int_\Sigma fLf e^{-\frac{|x|^2}{4}} d\mu < 0.$$

This shows that f is an unstable variation of Σ when Σ is considered as a minimal surface in the conformally changed metric. By definition, the dimension of the space of unstable variations of Σ as a minimal surface is the Morse index. This shows that the entropy index is bounded above by the Morse index. Combining this with Lemma 4.3 and Corollary 4.5 gives the desired bound.

Now we will show that the entropy index is at least $m - n - 2$. By definition of the Morse index, there exist m linearly independent eigenfunctions $u_1, \ldots, u_m \in C^{2,\alpha}(\Sigma)$ of L corresponding to the eigenvalues

$$\mu_1 < \mu_2 \leq \cdots \leq \mu_m < 0.$$

Colding and Minicozzi [10, Corollary 5.15] have shown that there exists an orthonormal basis of eigenfunctions of L for the weighted L^2 space. Therefore, without loss of generality, we may assume that $u_1, \ldots, u_m$ are orthonormal in the weighted L^2 space. The number of functions in $\{u_1, \ldots, u_m\}$ which are orthogonal to H and the components $\mathbf{n}_1, \ldots, \mathbf{n}_{n+1}$ of $\mathbf{n}$ is $m - n - 2$. Pick any such function u_i (if none exists, the entropy index is zero) and let Σ_ϵ be a variation of Σ by $u_i\mathbf{n}$. Choose any $y \in \mathbb{R}^{n+1}$ and any $h \in \mathbb{R}$ and set $x_\epsilon = \epsilon y$ and $t_\epsilon = 1 + \epsilon h$. Then

$$\begin{aligned}
(F_{x_\epsilon, t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} &= \int_\Sigma \left(-u_i L u_i - h^2 H^2 - \frac{1}{2}\langle y, \mathbf{n}\rangle^2 \right) e^{-\frac{|x|^2}{4}} d\mu \\
&\leq \mu_i \int_\Sigma u_i^2 e^{-\frac{|x|^2}{4}} d\mu \\
&< 0
\end{aligned}$$

by orthogonality and the assumption that $\mu_i < 0$. Since we had $m - n - 2$ possible linearly independent choices for such a u_i, the theorem follows. $\quad\square$

5 Entropy Instability of CSF Singularities

Having established in the previous section the relationship between the entropy index and Morse index of a shrinker, we will now calculate the Morse index of m-covered circles Γ_m and Abresch-Langer curves $\Gamma_{m,n}$. For 1-dimensional closed shrinkers, the Jacobi operator is Sturm-Liouville.

Proposition 5.1. *Let $\Gamma \subset \mathbb{R}^2$ be a shrinker. Then the Jacobi operator L of Γ is a Sturm-Liouville operator. In particular, for all $j \geq 1$, the eigenfunctions u_{2j-1} and u_{2j} of L have exactly $2j$ zeros.*

Proof. We refer the reader to [10, Section 5] for properties of the Jacobi operator of a shrinker. Differentiate the relation $k = \frac{1}{2}\langle x, \mathbf{n}\rangle$ to obtain $2k_s/k = \langle x, \mathbf{t}\rangle$ and note that $\nabla(\cdot) = \partial_s(\cdot)\mathbf{t}$. Therefore

$$
\begin{aligned}
L &= \Delta - \frac{1}{2}\langle x, \nabla(\cdot)\rangle + k^2 + \frac{1}{2} \\
&= \partial_{ss} - \frac{1}{2}\langle x, \partial_s(\cdot)\mathbf{t}\rangle + k^2 + \frac{1}{2} \\
&= \partial_{ss} - \frac{k_s}{k}\partial_s + k^2 + \frac{1}{2} \\
&= k\left(\partial_s\left(\frac{1}{k}\partial_s\right) + \frac{1}{k}\left(k^2 + \frac{1}{2}\right)\right).
\end{aligned}
$$

The latter part of the theorem follows from general Sturm-Liouville theory [7, Theorem 8.3.1]. $\quad\square$

For m-covered circles, the Jacobi operator reduces even further, as the curvature is constant. In this case, we are able to calculate the spectrum and the entropy unstable variations exactly (see Remark 5.3).

Theorem 5.2. *Let Γ_m be an m-covered circle of radius $\sqrt{2}$. Then the entropy index of Γ_m is*

$$
\Lambda(\Gamma_m) = 2\lceil\sqrt{2}m\rceil - 4,
$$

where $\lceil\,\cdot\,\rceil$ is the ceiling function. In particular, Γ_1 is the only entropy stable circle.

Proof. Since Γ_m has constant curvature, $k_s = 0$. Thus the Jacobi operator reduces to

$$
L = \partial_{ss} + 1,
$$

and the spectrum can be calculated explicitly:

$$
\mu_j = \frac{j^2}{2m^2} - 1
$$

for $j \in \mathbb{Z}$. Straightforward calculation using the above formula gives that the number of negative eigenvalues is $2\lceil\sqrt{2}m\rceil - 1$. The result now follows from Theorem 4.7. Note that the variation function corresponding to $j = 0$ is constant and hence proportional to k. Furthermore, the variation functions corresponding to $|j| = m$ are in the span of the component functions of the normal $\mathbf{n}$, since $\mu_j = -\frac{1}{2}$ in these cases. $\quad\square$

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Remark 5.3. The unstable variation functions of the m-covered circle can be determined explicitly. A unit speed parameterization $x : [0, 2\sqrt{2}\pi m] \to \mathbb{R}^2$ of the m-covered circle Γ_m of radius $\sqrt{2}$ is given by

$$x(\theta) = \sqrt{2}(\cos(\theta/\sqrt{2}), \sin(\theta/\sqrt{2})).$$

Straightforward calculation shows that the directions of unstable variations of Γ_m are given by the collection of functions $f_j, g_j : [0, 2\sqrt{2}\pi m] \to \mathbb{R}$ defined by

(5.4)
$$f_j(\theta) = \sin\left(\frac{j}{\sqrt{2m}}\theta\right) \qquad \text{and} \qquad g_j = \cos\left(\frac{j}{\sqrt{2m}}\theta\right),$$

for $j \in \mathbb{N}$ such that $1 \leq j < \sqrt{2}m$ and $j \neq m$. By Theorem 5.2, these are all of the entropy unstable variations. Geometrically, varying Γ_m by these functions corresponds to enlarging some circles of Γ_m, while contracting the others. The CSF will amplify this perturbation.

The proof of Theorem A.2 now follows easily using Theorem 5.2.

Proof of Theorem A.2. Theorem 5.2 shows that an m-covered circle is entropy unstable for $m \geq 2$. By Theorem 4.2, the m-covered circle Γ_m can be perturbed to a curve Γ' with lower entropy. Moreover, since the perturbation can be chosen to be arbitrarily C^∞-small, the curvature of Γ' can be chosen close enough to the curvature of Γ_m so that the perturbed curve also has turning number m. The second part of Theorem A.2 follows immediately from Theorem A.1. $\qquad\square$

Entropy instability means that we can perturb a shrinker slightly to decrease entropy, and thus that the singularity corresponding to the given shrinker will never occr along the flow starting from the perturbed shrinker. In Section 6, our construction of a piecewise CSF for closed immersed curves, whose only singularities are embedded circles and type II singularities, rests upon the fact that the embedded circle is the only entropy stable closed plane shrinker. The latter fact will follow from

Theorem 5.5. *An Abresch-Langer curve $\Gamma_{m,n}$ has entropy index*

(5.6)
$$\Lambda(\Gamma_{m,n}) = 2n - 5.$$

Proof. Let L be the Jacobi operator of $\Gamma := \Gamma_{m,n}$. We claim that k_s/k is an eigenfunction of L with eigenvalue 0, i.e. that $L(k_s/k) = 0$. Indeed, differentiating the shrinker equation (1.8) gives $2k_s/k = \langle x, \mathbf{t} \rangle$. Therefore

$$\begin{aligned}
2L(k_s/k) &= \partial_{ss}\langle x, \mathbf{t} \rangle - \frac{k_s}{k}\partial_s\langle x, \mathbf{t} \rangle + (k^2 + \frac{1}{2})\langle x, \mathbf{t} \rangle \\
&= \partial_s(1 - k\langle x, \mathbf{n} \rangle) - \frac{1}{2}\langle x, \mathbf{t} \rangle(1 - k\langle x, \mathbf{n} \rangle) + (k^2 + \frac{1}{2})\langle x, \mathbf{t} \rangle \\
&= -k_s\langle x, \mathbf{n} \rangle - k^2\langle x, \mathbf{t} \rangle + \frac{1}{2}k\langle x, \mathbf{t} \rangle\langle x, \mathbf{n} \rangle + k^2\langle x, \mathbf{t} \rangle \\
&= -k_s\langle x, \mathbf{n} \rangle + k_s\langle x, \mathbf{n} \rangle \\
&= 0.
\end{aligned}$$

Since k is strictly positive and k_s has exactly $2n$ zeros on $[0, 2\pi m)$, k_s/k is either the $(2n-1)^{\text{st}}$ or $2n^{\text{th}}$ eigenfunction of L. It remains to show that k_s/k is the $(2n-1)^{\text{st}}$ eigenfunction and that zero is a simple eigenvalue. To do so, we reparameterize Γ using the variable

$$\theta = -\cos^{-1}\langle e_1, \mathbf{n} \rangle,$$

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where $e_1 \in \mathbb{R}^2$ is a constant unit vector. This is possible since Γ is convex (however, θ is only locally continuous as a discontinuity appears after one round). It follows from the shrinker equation that if we define the operator $\widetilde{L}$ by

$$\widetilde{L} = k^2 \partial_{\theta\theta} + (k^2 + 1/2),$$

then $\frac{\partial\theta}{\partial s} = k$ and $\widetilde{L}f = Lf$. The theorem now follows from the linear analysis in [11, Proposition 2.1], however, we provide a proof here for the convenience of the reader.

Consequently, $k_\theta = k_s/k$ and

$$\widetilde{L}k_\theta = \widetilde{L}(k_s/k) = L(k_s/k) = 0.$$

In other words, k_θ is an eigenfunction of $\widetilde{L}$ with eigenvalue 0. We denote by $\{\mu_j\}$ and $\{\nu_j\}$, respectively, the Dirichlet and Neumann eigenvalues of $\widetilde{L}$ on $[0, \pi m/n]$. Let f_j solve the Neumann problem $\widetilde{L}f_j = -\nu_j f_j$ on $[0, \pi m/n]$. Since k is an even function, f_j may be extended by odd reflection and then periodically to an eigenfunction of $\widetilde{L}$ on $[0, 2\pi m]$ with the same eigenvalue ν_j. The eigenfunction f_2 will have exactly $2n$ zeros on $[0, 2\pi m)$ and k_θ is the lowest Dirichlet eigenvalue of $\widetilde{L}$ on $[0, \pi m/n]$, so $\mu_1 = 0$. From the standard fact that $\mu_1 \geq \nu_2$, it follows that k_θ and f_2 are, respectively, the $(2n-1)^{\text{st}}$ and $2n^{\text{th}}$ eigenfunctions of $\widetilde{L}$ on $[0, 2\pi m]$. Moreover, since $\widetilde{L}$ and L have the same eigenfunctions $k_s/k = k_\theta$ is the $(2n-1)^{\text{st}}$ eigenfunction of L.

Next we show that zero is a simple eigenvalue. Note that $\widetilde{L}$ is a linear second order differential operator so the equation $\widetilde{L}f = 0$ can have at most two linearly independent solutions. If zero were not a simple eigenvalue, then there would exist a nontrivial $2\pi m$ periodic solution w of $\widetilde{L}w = 0$. Consequently, any solution of $\widetilde{L}f = 0$, being a linear combination of k_θ and w, which are both $2\pi m$ periodic, would have to be $2\pi m$ periodic. As a result, to show that zero is a simple eigenvalue, it is sufficient to produce a solution to $\widetilde{L}f = 0$ which is not $2\pi m$ periodic.

We will now produce such a solution. It follows from the shrinker equation that k solves the ODE

(5.7)
$$k_{\theta\theta} + k - \frac{1}{2k} = 0.$$

This equation has first integral

$$E = k_\theta^2 + k^2 - \frac{1}{4}\log k$$

and the general solution can be expressed as $k(\theta + a, E)$. Let

$$u = \frac{\partial k}{\partial E}\bigg|_{E=E_{m,n}},$$

where $E_{m,n}$ is the constant corresponding to $\Gamma_{m,n}$. Straightforward calculation using (5.7) shows that $\widetilde{L}u = 0$. Abresch-Langer [1, Proposition 3.2] show that u is not $2\pi m$ periodic. As stated above, it follows that zero is a simple eigenvalue of $\widetilde{L}$ and thus also of L.

Summarizing, we have shown that $L(k_s/k) = 0$, that k_s/k is the $(2n-1)^{\text{st}}$, and that zero is a simple eigenvalue of L. Consequently, L has $2n-2$ negative eigenvalues. By Theorem 4.7, the Abresch-Langer curve $\Gamma_{m,n}$ has entropy index $2n-5$. $\qquad\square$

Corollary 5.8. *The embedded circle is the only entropy stable closed singularity of the CSF for closed curves.*

6 Generic CSF

In this section, we prove Theorem B. The following lemma shows that, as a consequence of Corollary 5.8, we can perturb an unstable closed plane shrinker in such a way as to satisfy the entropy inequality (1.7) for a piecewise CSF. For the piecewise flow in Theorem B, we will actually require the entropy inequality (1.7) to be a strict inequality so that we can exclude a shrinker from appearing as a singularity at later times of the flow.

Lemma 6.1. *Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve such that the CSF $(\Gamma_t)_{t\in[0,T)}$ starting at Γ has a type I singularity other than the embedded circle and let $\widetilde{\Gamma}_\infty \subset \mathbb{R}^2$ be some blowup sequence limit. Then there exists a $T_0 > 0$ and a function $u \in C^\infty(\Gamma)$ such that the graph $\bar{\Gamma} = \Gamma_{T_0} + u\mathbf{n}$ satisfies*

$$(6.2) \qquad\qquad \lambda(\bar{\Gamma}) < \lambda(\widetilde{\Gamma}_\infty).$$

Proof. If the CSF $(\Gamma_t)_{t\in[0,T)}$ has a type I singularity at time T, any limit of a rescaling sequence is a closed shrinker and thus is compact. Therefore, there is only one singular point $x_1 \in \mathbb{R}^2$ at time T. Without loss of generality, we may assume that $0 \in \mathbb{R}^2$ is a singular point. By the classification of 1-dimensional shrinkers [1], $\widetilde{\Gamma}_\infty$ must be a multiply-covered circle or a multiply-covered Abresch-Langer curve. Since we assume that the shrinker $\widetilde{\Gamma}_\infty$ is not an embedded circle, Corollary 5.8 implies that $\widetilde{\Gamma}_\infty$ is entropy unstable. By Theorem 2.3, there exists a variation $f \in C^\infty(\widetilde{\Gamma}_\infty)$ and an $\epsilon_0 > 0$ such that

$$\lambda(\widetilde{\Gamma}_\infty + \epsilon f\mathbf{n}) < \lambda(\widetilde{\Gamma}_\infty),$$

for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_0$. By assumption, there exists a sequence $\tau_i \to \infty$ of rescaled times such that the rescaled curves $(\widetilde{\Gamma}_{\tau_i})_{i\in\mathbb{N}}$ smoothly converge to $\widetilde{\Gamma}_\infty$. Thus, there exists $N' \in \mathbb{N}$ and a sequence $(g_i)_{i\in\mathbb{N}} \subset C^\infty(\widetilde{\Gamma}_\infty)$ with $\|g_i\|_{C^\infty} \to 0$ such that, for $i > N'$, the rescaled curve $\widetilde{\Gamma}_{\tau_i}$ can be written as a graph

$$\widetilde{\Gamma}_{\tau_i} = \widetilde{\Gamma}_\infty + g_i\mathbf{n}.$$

By the monotonicity of entropy and Corollary 2.5, there exists $\epsilon_1 \leq \epsilon_0$ and $N \geq N'$ such that, for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_1$ and all $i > N$, we have

$$\lambda(\widetilde{\Gamma}_\infty + (\epsilon f + g_i)\mathbf{n}) < \lambda(\widetilde{\Gamma}_\infty) \leq \lambda(\widetilde{\Gamma} + g_i\mathbf{n}).$$

Since entropy is invariant under dilations, when these curves are rescaled back to original space-time, the entropy condition can be satisfied when $|\epsilon| \neq 0$ is small enough and i is large enough. The time T_0 then corresponds to the rescaled time τ_i and the function u is a multiple of f. $\qquad\square$

Using the classification of singularities for the CSF of closed curves and Lemma 6.1, we now prove Theorem B.

Proof of Theorem B. We will construct a piecewise CSF with a finite number of discontinuities that eventually becomes extinct in as an embedded circle, or has a type II singularity. We perform a smooth jump just before a (entropy unstable) singular time, where we replace a time slice of the flow by a graph over it, and the crucial point is to show that the entropy decreases below the entropy of the unstable singularity. We repeat this until we get a singular point where every rescaled singularity is either an embedded circle or a type II singularity

Let $(\Gamma_t)_{t\in[0,T_1)}$ be the CSF starting at Γ. The CSF $(\Gamma_t)_{t\in[0,T_1)}$ has either a type I or type II singularity at time T_1. We consider both cases:

Case (I): If the CSF $(\Gamma_t)_{t \in [0, T_1)}$ has a type I singularity at time T_1, any limit of a rescaling sequence is either an entropy stable or entropy unstable closed shrinker. In particular, any limit of a rescaling sequence is compact, and therefore, there is only one singular point $x_1 \in \mathbb{R}^2$ at time T_1.

Case (I.a): If there is some rescaling sequence limit which is an entropy unstable closed shrinker Γ^1_∞, Lemma 6.1 gives the existence of a $t_2 < T_1$ and a curve $\Gamma^2_{t_2}$ such that $\Gamma^2_{t_2}$ is a graph over $\Gamma^1_{t_2} := \Gamma_{t_2}$ of a function $u_2 \in C^{2,\alpha}(\Gamma^1_{t_2})$ satisfying

$$\lambda(\Gamma^2_{t_2}) < \lambda(\Gamma^1_\infty).$$

Consequently, there exists a CSF $(\Gamma^2_t)_{t \in [t_2, T_2)}$ starting at $\Gamma^2_{t_2}$ such that the concatenation of $(\Gamma^1_t)_{t \in [0, t_2]}$ with $(\Gamma^2_t)_{t \in [t_2, T_2)}$ is a piecewise CSF starting at Γ.

Case (I.b): If all rescaling sequences converge to an entropy stable closed shrinker, it must be an embedded circle, so the theorem holds.

Case (II): If the CSF $(\Gamma_t)_{t \in [0, T_1)}$ has a type II singularity at time T_1, the theorem holds.

If *Case (I.a)* applies, we return to cases *(I)* and *(II)*, this time applied to $(\Gamma^2_t)_{t \in [t_2, T_2)}$, and repeat the process. Since piecewise CSF preserves turning number and since there are only finitely many closed plane shrinkers of any given turning number, it follows immediately from the fact that

$$\lambda(\Gamma^i_{T_i}) < \lambda(\Gamma^{i-1}_\infty)$$

at each step and from the monotonicity of entropy under piecewise CSF that *Case (I.a)* can apply only finitely many times. This proves the first part of the theorem.

It was shown in the proof of Theorem A.1 that any blowup sequence of a type I singularity of the CSF for closed curves preserves turning number. It follows that a closed curve with turning number greater than 1 cannot have an embedded circle as the limit of a rescaling sequence. This proves the second part of Theorem B. $\qquad\square$

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