Positive Traces on Deformations of Kleinian Singularities of Type D

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Abstract

Filtered deformations of Kleinian singularities have received much attention over the last half a century, particularly for their importance to various areas of algebra such as representation theory and Lie theory. Recently, traces on filtered deformations of Kleinian singularities have been studied for their connection with star-products, associative products with significance in algebra and theoretical physics, namely superconformal field theory. We build on a line of work investigating positive traces on deformations of Kleinian singularities. In a special case, we find analogues of classification theorems of traces on deformations of Kleinian singularities of type A for Kleinian singularities of type D and prove that positive traces of type D are restrictions of positive traces of type A.

Summary

Kleinian singularities are important algebraic structures with connections to several fields in algebra and physics. Functions known as traces on algebras related to Kleinian singularities are useful in theoretical physics, particularly superconformal field theory, in additional to being of algebraic interest. We study positive traces, which are especially physically meaningful and algebraically significant. We classify positive traces in a special case for an important class of Kleinian singularities, particularly those of type D, building on previous work in the type A case.

1 Introduction

A Kleinian singularity is a quotient $\mathbb{C}^2/G$, where G is a finite subgroup of the special linear group $\mathrm{SL}_2(\mathbb{C})$. The Kleinian singularity $\mathbb{C}^2/G$ has as its coordinate ring the commutative graded algebra $A = \mathbb{C}[u,v]^G$ of G-invariant polynomials in two variables. Kleinian singularities have been extensively studied and are significant in many areas of mathematics, such as algebraic geometry and representation theory [2, 3, 4].

We consider filtered deformations $\mathcal{A}$ of A, that is, filtered associative algebras $\mathcal{A}$ with associated graded algebra $\operatorname{gr}\mathcal{A}$ equal to A. Deformations of Kleinian singularities have been the subject of a large body of research which has uncovered numerous connections between such deformations and areas of algebra such as Lie theory; see [1] for a brief overview. More recently, filtered deformations of algebras have been studied in connection with star-products, which are associative products on the algebra with significance in algebra in superconformal field theory, with a special focus on the relationship between nondegenerate short star-products of an algebra A and nondegenerate twisted traces of the algebra's filtered deformation $\mathcal{A}$ [5, 6].

For a filtered algebra B and a filtration-preserving invertible linear map g, a *g-twisted trace* on B is a linear map $T\colon B \to \mathbb{C}$ such that $T(ab) = T(bg(a))$ for all $a, b \in B$. Etingof and Stryker [5] proved that nondegenerate short star-products of an algebra A correspond to nondegenerate twisted traces of the filtered deformation $\mathcal{A}$ of A. Twisted traces on deformations of Kleinian singularities of type A, that is, Kleinian singularities with G a cyclic group, were subsequently classified by Etingof, Klyuev, Rains, and Stryker [6].

Positive traces are of particular physical and algebraic interest. Given an algebra B and an antilinear automorphism $\rho\colon B \to B$ such that $\rho^2 = \mathrm{id}$, a trace T on B is said to be *positive* if $T(a\rho(a)) > 0$ for all nonzero a in B. Positive traces are in correspondence with positive definite Hermitian forms on B such that $(ab, c) = (a, \rho(b)c)$ for all a, b, c in B, and positive traces of filtered deformations correspond to unitary star-products, which are of particular physical interest; see [6] for a summary of research on positive traces. Positive traces were characterized by Etingof, Klyuev, Rains, and Stryker [6] for Kleinian singularities of type A.

In this paper, we study traces of filtered deformations of Kleinian singularities of type D, that is, Kleinian singularities with G a dicyclic group. Filtered deformations of Kleinian singularities depend on an element c of the center $Z(\mathbb{C}[G])$ of the corresponding group algebra. Under an assumption on this parameter c, described in Section 2, we characterize traces, and under an additional assumption about the structure of the filtered deformation, described in Section 5, we characterize positive traces. Our main result is that every positive trace on a filtered deformation of a Kleinian singularity of type D satisfying this set of assumptions is the restriction of a positive trace on a filtered deformation of a Kleinian singularity of type A.

We begin by outlining the technical preliminaries for our work in Section 2. Then, in Section 3, we introduce several algebra elements which are important for our results and prove various identities and relations among them. We characterize traces of deformations of Kleinian singularities of type D under an assumption on the element c in Section 4, and use this characterization to obtain an analytic formula for traces in a special case in Section 5.

Finally, in Section 6, we prove our main result regarding positive traces.

2 Preliminaries

Throughout the paper, we fix a positive even integer n with $m = \frac{n}{2}$ and denote $\varepsilon = \exp\left(\frac{2\pi i}{n}\right)$. For a finite subgroup G of the special linear group $\mathrm{SL}_2(\mathbb{C})$, the Kleinian singularity $\mathbb{C}^2/G$ has as its coordinate ring the commutative graded algebra $A = \mathbb{C}[u,v]^G$ of G-invariant polynomials in two variables, with the action of G on A defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot P(u,v) = P(au+bv, cu+dv).$$

We are interested in the case when $G = \mathbb{D}_n$, the dicyclic group generated by elements

$$g = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \qquad h = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with $g^n = h^4 = ghgh^{-1}$ all equal to the group identity and $g^m = h^2$. The group $\mathbb{D}_n$ acts on $\mathbb{C}[u,v]$ by $g \cdot u = \varepsilon u$, $g \cdot v = \varepsilon^{-1} v$, $h \cdot u = v$, $h \cdot v = -u$, and the invariant polynomials generating A are u^2v^2, $u^n + v^n$, and $uv(u^n - v^n)$.

All elements of the algebra A are the sum of homogeneous polynomials, so A is graded by the degree of homogeneous polynomials. We consider filtered deformations $\mathcal{A}$ of A, that is, filtered associative algebras $\mathcal{A}$ with associated graded algebra $\operatorname{gr}\mathcal{A}$ equal to A. In particular, we make use of the construction of filtered deformations presented by Crawley-Boevey and Holland [1]. Let $\mathbb{C}[x,y]\#G$ be the skew group algebra constructed from the vector space $\mathbb{C}[x,y] \otimes \mathbb{C}[G]$ with multiplication given by

$$(P \otimes g)(G \otimes h) = Pg(Q) \otimes gh,$$

where P and Q are polynomials, g and h are group elements, and $g(Q)$ denotes the action of g on Q. For an element c of the center $Z(\mathbb{C}[G])$ of the group algebra, let

$$H_c = (\mathbb{C}[x,y]\#G)/(xy - yx - c).$$

For the idempotent $e = \frac{1}{|G|}\sum_{g \in G} g$, let $\mathcal{O}_c = eH_ce$, which is an algebra with unit e. Losev [7, Theorem 3.4] proved that for any finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$, any filtered deformation of $\mathbb{C}[x,y]^G$ is isomorphic to some algebra $\mathcal{O}_c$ with $c \in Z(\mathbb{C}[g])$.

A trace on $\mathcal{A}$ is a linear function $T\colon \mathcal{A} \to \mathbb{C}$ satisfying $T(ab) = T(ba)$ for all $a, b \in \mathcal{A}$. We only consider traces on noncommutative algebras, so by [1, Theorem 0.4(1)], we may assume the coefficient of c on the group identity is nonzero. Also, Crawley-Boevey and Holland [1, Theorem 0.4(2)] proved that for c outside a finite set of hyperplanes, any trace on $\mathcal{O}_c$ is a restriction of a trace on H_c. We consider only such c, which we call *generic*. For a classification of traces on $\mathcal{O}_c$, it therefore suffices to characterize traces on H_c.

Letting C_n denote the cyclic group of n elements, we further assume that c is in $\mathbb{C}[C_n]$. Then in the algebra H_c, the following hold:

- $gx = \varepsilon xg$;
- $gy = \varepsilon^{-1} yg$;
- $hx = yh$;
- $hy = -xh$;
- $xy - yx = c = \sum_{i=0}^{n-1} c_i g^i$, with $c_0 \neq 0$;
- $c_i = c_{-i}$ for all i.

Here the equality $c_i = c_{-i}$ holds because c is in $Z(\mathbb{C}[\mathbb{D}_n])$ and the conjugacy classes of $\mathbb{D}_n$ are of the form $\{g^i, g^{-i}\}$, $\{g^{2i}h : i \in \mathbb{Z}\}$, and $\{g^{2i+1}h : i \in \mathbb{Z}\}$.

We close this section with a definition of positive traces. For an algebra B and an antilinear automorphism $\rho\colon B \to B$ such that $\rho^2 = \mathrm{id}$, a trace T on B is said to be *positive* if $T(a\rho(a)) > 0$ for all nonzero a in B. The antilinear automorphism we study is defined on H_c by $\rho(x) = y$, $\rho(y) = -x$, $\rho(g) = g$, $\rho(h) = h$. We return to positive traces in Section 6.

3 Algebra Identities and Relations

In this section, we introduce several important algebra elements and describe relations between them.

Recall that $c = \sum_{i=0}^{n-1} c_i g^i$, with g a generator of $\mathbb{D}_n$ such that g^n is the group identity, and $\varepsilon = \exp\left(\frac{2\pi i}{n}\right)$. Let

$$k = \frac{1}{c_0}\left(xy + \sum_{i=1}^{n-1} \frac{c_i g^i}{\varepsilon^i - 1}\right) - \frac{1}{2}.$$

The following lemma describes commutativity relations involving k which are important for our results.

Lemma 3.1. *The following equalities hold in H_c:*

- $gk = kg$;
- $hk = -kh$;
- $[k, x] = -x$;
- $kx = x(k-1)$;
- $[k, y] = y$;
- $ky = y(k+1)$.

Proof. Because $gxy = \varepsilon xgy = xyg$, it is clear that g and k commute. Also, we have

$$\begin{aligned}
hk &= \frac{1}{c_0}\left(hxy + \sum_{i=1}^{n-1} \frac{hc_i g^i}{\varepsilon^i - 1}\right) - \frac{h}{2} \\
&= \frac{1}{c_0}\left(-yxh + \sum_{i=1}^{n-1} \frac{c_i g^{-i} h}{\varepsilon^i - 1}\right) - \frac{h}{2} \\
&= \frac{1}{c_0}\left(c - xy + \sum_{i=1}^{n-1} \frac{c_i g^i}{\varepsilon^{-i} - 1}\right) h - \frac{h}{2} \\
&= \frac{1}{c_0}\left(-xy + c_0 + \sum_{i=1}^{n-1} \left(c_i g^i - \frac{\varepsilon^i c_i g^i}{\varepsilon^i - 1}\right)\right) h - \frac{h}{2} \\
&= \frac{1}{c_0}\left(-xy + c_0 - \sum_{i=1}^{n-1} \frac{c_i g^i}{\varepsilon^i - 1}\right) h - \frac{h}{2} \\
&= -\frac{1}{c_0}\left(xy + \sum_{i=1}^{n-1} \frac{c_i g^i}{\varepsilon^i - 1}\right) h + \frac{h}{2} \\
&= -kh.
\end{aligned}$$

We now prove the commutation relations with x and y. We have $xyx = x(xy - c) = x^2y - xc$, so

$$\begin{aligned}
[k, x] &= \frac{1}{c_0}\left(xyx - x^2y + \sum_{i=1}^{n-1} \frac{c_i g^i x - x c_i g^i}{\varepsilon^i - 1}\right) \\
&= \frac{1}{c_0}\left(-xc + \sum_{i=1}^{n-1} \frac{\varepsilon^i x c_i g^i - x c_i g^i}{\varepsilon^i - 1}\right) \\
&= \frac{1}{c_0}\left(-xc + x\sum_{i=1}^{n-1} c_i g^i\right) \\
&= -x.
\end{aligned}$$

It follows immediately that $kx = x(k-1)$, and the commutation relations with y follow by conjugating the commutation relations with x by h. □

Now let

$$e_q = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{iq} g^i$$

for integers q. Note that $e_q = e_{q+n}$ for all q. The elements e_q form an important basis for $\mathbb{C}[C_n]$, and it is often convenient to express elements of H_c using e_q. The following lemma describes relations involving e_q.

Lemma 3.2. *The following equalities hold in H_c:*

- $e_q g^i = g^i e_q = \varepsilon^{-iq} e_q$;
- $h e_q = e_{-q} h$;
- $e_q x = x e_{q+1}$;
- $e_q y = y e_{q-1}$;
- $e_q k = k e_q$;
- $e_p e_q = \delta_{p,q} e_q$, *where δ is the Kronecker delta.*

Proof. Clearly g^i and e_q commute. We have

$$g^i e_q = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon^{jq} g^{i+j} = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon^{(j-i)q} g^j = \varepsilon^{-iq} e_q$$

and

$$h e_q = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{iq} h g^i = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{iq} g^{-i} h = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{-iq} g^i h = e_{-q} h,$$

proving the commutation relations with group elements. Also,

$$e_q x = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{iq} g^i x = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{iq+i} x g^i = \frac{1}{n} x \sum_{i=0}^{n-1} \varepsilon^{i(q+1)} g^i = x e_{q+1},$$

and conjugating by h, we obtain $e_q y = y e_{q-1}$. It follows from the established commutation relations that e_q commutes with xy and each g^i, so e_q commutes with k. Finally, we have

$$e_p e_q = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{ip} g^i e_q = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon^{i(p-q)} e_q.$$

If $p = q$, then all terms in the sum are equal to e_q, so $e_p e_q = e_q$. If $p \neq q$, then $\sum_{i=0}^{n-1} \varepsilon^{i(p-q)} = 0$, so $e_p e_q = 0$. □

Finally, we define

$$\alpha_q = \frac{1}{c_0}\sum_{i=1}^{n-1} \frac{c_i \varepsilon^{-iq}}{\varepsilon^i - 1}$$

for integers q, and once again observe that $\alpha_q = \alpha_{q+n}$ for all q. The elements α_q have an important symmetry and allow for useful transitions from complicated expressions involving e_q to scalar multiples of e_q, as shown by the following lemma.

Lemma 3.3. *The following statements hold in H_c:*

- $\alpha_q = -\alpha_{-q-1}$;
- *The elements* α_p *and* e_q *satisfy*

$$\alpha_p e_q = \frac{1}{c_0}\left(\sum_{i=1}^{n-1} \frac{c_i \varepsilon^{i(q-p)} g^i}{\varepsilon^i - 1}\right) e_q.$$

Proof. We have

$$\begin{aligned}\alpha_q &= \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{-iq}}{\varepsilon^i-1} = \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_{-i}\varepsilon^{-iq}}{\varepsilon^i-1} = \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{iq}}{\varepsilon^{-i}-1}\\ &= \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{i(q+1)}}{1-\varepsilon^i} = -\alpha_{-q-1}.\end{aligned}$$

We also have

$$\begin{aligned}\alpha_p e_q &= \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{-ip}e_q}{\varepsilon^i-1}\\ &= \frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{i(q-p)}\varepsilon^{-iq}e_q}{\varepsilon^i-1}\\ &= \frac{1}{c_0}\left(\sum_{i=1}^{n-1}\frac{c_i\varepsilon^{i(q-p)}g^i}{\varepsilon^i-1}\right)e_q,\end{aligned}$$

proving the lemma. □

The elements e_q are a basis for $\mathbb{C}[g]$, so we may write

$$k = \frac{1}{c_0}\left(xy - \sum_{q=0}^{n-1}\beta_q e_q\right), \quad \text{or} \quad xy = c_0 k + \sum_{q=0}^{n-1}\beta_q e_q,$$

for complex numbers β_q and extend the β_q modulo n, so that $\beta_q = \beta_{q+n}$ for all integers q. More generally, we have the following lemma.

Lemma 3.4. *For all positive integers* a*, the elements* $x^a y^a$ *and* $y^a x^a$ *are polynomials in* k *and group elements given by*

$$\begin{aligned}x^a y^a &= \prod_{i=0}^{a-1}\left(c_0(k+i) + \sum_{q=0}^{n-1}\beta_{q+i}e_q\right),\\ y^a x^a &= (-1)^a\prod_{i=0}^{a-1}\left(c_0(-k+i) + \sum_{q=0}^{n-1}\beta_{q+i}e_{-q}\right).\end{aligned}$$

Also, the elements $x^a y^a e_q$ and $y^a x^a e_q$ can be written as polynomial expressions in k multiplied by e_q as follows:

$$x^a y^a e_q = e_q \prod_{i=0}^{a-1} (c_0(k+i) + \beta_{q+i}),$$
$$y^a x^a e_q = (-1)^a e_q \prod_{i=0}^{a-1} (c_0(-k+i) + \beta_{-q+i}).$$

Proof. We prove by induction the polynomial expression for $x^a y^a$ by induction, with the base case of $a = 1$ true by definition. Assume that $x^a y^a e_q = e_q \prod_{i=0}^{a-1}(c_0(k+i) + \sum_{q=0}^{n-1} \beta_{q+i} e_q)$. Then

$$\begin{aligned}
x^{a+1}y^{a+1} &= x^a(xy)y^a \\
&= x^a \left(c_0 k + \sum_{q=0}^{n-1} \beta_q e_q \right) y^a \\
&= x^a y^a \left(c_0(k+a) + \sum_{q=0}^{n-1} \beta_q e_{q-a} \right) \\
&= x^a y^a \left(c_0(k+a) + \sum_{q=0}^{n-1} \beta_{q+a} e_q \right) \\
&= \left(\prod_{i=0}^{a-1} \left(c_0(k+i) + \sum_{q=0}^{n-1} \beta_{q+i} e_q \right) \right) \left(c_0(k+a) + \sum_{q=0}^{n-1} \beta_{q+a} e_q \right) \\
&= \prod_{i=0}^{a} \left(c_0(k+i) + \sum_{q=0}^{n-1} \beta_{q+i} e_q \right),
\end{aligned}$$

as needed. The analogous claim for $y^a x^a$ follows from conjugation by h. The expressions for $x^a y^a e_q$ and $y^a x^a e_q$ follow using the statement that $e_p e_q = \delta_{p,q} e_q$ from Lemma 3.2. □

The numbers β_q are related to the α_q by the following lemma.

Lemma 3.5. *For all q, we have $\beta_q = \frac{c_0}{2} - c_0 \alpha_q$.*

Proof. By Lemma 3.3, we have

$$\begin{aligned}
xye_q &= \left(c_0 k + \frac{c_0}{2} - \sum_{i=1}^{n-1} \frac{c_i g^i}{\varepsilon^i - 1} \right) e_q \\
&= \left(c_0 k + \frac{c_0}{2} - c_0 \alpha_q \right) e_q.
\end{aligned}$$

Also, by definition, $xye_q = (c_0 k + \beta_q)\, e_q$. Thus

$$(c_0 k + \beta_q)\, e_q = \left(c_0 k + \frac{c_0}{2} - c_0\alpha_q\right) e_q$$
$$\beta_q e_q = \left(\frac{c_0}{2} - c_0\alpha_q\right) e_q,$$

whence it follows that $\beta_q = \frac{c_0}{2} - c_0\alpha_q$. □

4 Characterization of Traces

In this section, we study traces using basis elements of the form $x^i R(k) e_q$, $y^i R(k) e_q$, $x^i R(k) e_q h$, and $y^i R(k) e_q h$, with $R(k) \in \mathbb{C}[k]$. We begin with a lemma about the values $T(e_q h)$.

Lemma 4.1. *If $m \nmid q$, then the value $T(e_q h)$ is equal to* 0.

Proof. Because the value of T must be equal on all conjugacy classes of $\mathbb{D}_n$, we have

$$\begin{aligned} T(e_q h) &= \frac{1}{n} T\left(\sum_{i=0}^{n-1} \varepsilon^{iq} g^i h\right) \\ &= \frac{1}{n}\left(\sum_{i=0}^{m-1} \varepsilon^{2iq}\right) T(h) + \frac{1}{n}\left(\sum_{i=0}^{m-1} \varepsilon^{(2i+1)q}\right) T(gh) \\ &= \frac{1}{n}\left(\sum_{i=0}^{m-1} \varepsilon^{2iq}\right) T(h) + \frac{1}{n}\varepsilon^{q}\left(\sum_{i=0}^{m-1} \varepsilon^{2iq}\right) T(gh). \end{aligned}$$

If $m \nmid q$, then $n \nmid 2q$, so the sums of roots of unity are equal to 0. Therefore, $T(e_q h)$ is nonzero only if $m \mid q$. □

The following theorem classifies traces on H_c. In particular, Conditions (1)–(6) express trace values on H_c in terms of trace values on elements of the form $R(k)e_q$, with R a polynomial, and Conditions (7) and (8) restrict possible trace values on such terms.

Theorem 4.2. *A linear map $T : H_c \to \mathbb{C}$ is a trace on H_c if and only if all of the following conditions hold:*

(1) *T is invariant under conjugation by elements of $\mathbb{D}_n$;*

(2) *$T(x^i R(k) e_q) = T(y^i R(k) e_q) = 0$ when $i \neq 0$;*

(3) *$T(x^i R(k) e_q h) = T(y^i R(k) e_q h) = 0$ when $2 \nmid i$;*

(4) *$T(x^{2i} R(k) e_q h) = T(S(k) R(k+i) e_{q-i} h)$, where S is the polynomial satisfying that $x^i y^i e_{q-i} = S(k) e_{q-i}$;*

(5) $T(y^{2i}R(k)e_qh) = (-1)^iT(S(k)R(k-i)e_{q+i}h)$, *where* S *is the polynomial satisfying that* $y^ix^ie_{q+i} = S(k)e_{q+i}$*;*

(6) $T(k^ie_qh) = 0$ *for* $i > 0$*;*

(7) $T((k+\frac{1}{2}-\alpha_q)R(k+\frac{1}{2})e_q) = T((k-\frac{1}{2}-\alpha_q)R(k-\frac{1}{2})e_{q+1})$ *for all* $q \in \mathbb{Z}$ *and* $R \in \mathbb{C}[X]$*;*

(8) $T(R(k)e_q) = T(R(-k)e_{-q})$ *for all* $q \in \mathbb{Z}$ *and* $R \in \mathbb{C}[X]$*.*

Proof. Assume first that T is a trace; we will show that the listed conditions must hold.

(1) This is part of the definition of a trace.

(2) The trace condition gives $0 = T([k, x^iR(k)e_q]) = -iT(x^iR(k)e_q)$, and similarly $0 = T([k, y^iR(k)e_q]) = iT(y^iR(k)e_q)$, from which the claim follows.

(3) Conjugation by h^2 multiplies both x and y by -1 while fixing e_q and h, so $T(x^iR(k)e_qh) = (-1)^iT(x^iR(k)e_qh)$ and $T(y^iR(k)e_qh) = (-1)^iT(y^iR(k)e_qh)$, proving the claim.

(4) Using the trace condition to commute x^i and $x^iR(k)e_qh$, we obtain

$$\begin{aligned} T(x^{2i}R(k)e_qh) &= T(x^iR(k)e_qhx^i) \\ &= T(x^iR(k)e_qy^ih) \\ &= T(x^iy^iR(k+i)e_{q-i}h) \\ &= T(x^iy^ie_{q-i}R(k+i)h) \\ &= T(S(k)e_{q-i}R(k+i)h) \\ &= T(S(k)R(k+i)e_{q-i}h), \end{aligned}$$

as desired.

(5) Similarly to the proof of Condition (4), we obtain

$$\begin{aligned} T(y^{2i}R(k)e_qh) &= T(y^iR(k)e_qhy^i) \\ &= (-1)^iT(y^iR(k)e_qx^ih) \\ &= (-1)^iT(y^ix^iR(k-i)e_{q+i}h) \\ &= (-1)^iT(S(k)R(k-i)e_{q+i}h), \end{aligned}$$

as needed.

(6) The trace condition gives $0 = T([k, k^{i-1}e_qh]) = T(2k^ie_qh)$, from which the claim follows.

(7) Using Lemma 3.3 and the commutativity of e_q and k, we obtain

$$
\begin{aligned}
T\left(\left(k+\tfrac{1}{2}-\alpha_q\right)R\left(k+\tfrac{1}{2}\right)e_q\right) &= T\left(\left(k+\frac{1}{2}-\frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i g^i}{\varepsilon^i-1}\right)R\left(k+\tfrac{1}{2}\right)e_q\right)\\
&= \frac{1}{c_0}T\left(xyR\left(k+\tfrac{1}{2}\right)e_q\right).
\end{aligned}
$$

Using the trace condition to commute x and $yR(k+\frac{1}{2})e_q$, we find

$$
\begin{aligned}
T\left(\left(k+\tfrac{1}{2}-\alpha_q\right)R\left(k+\tfrac{1}{2}\right)e_q\right) &= \frac{1}{c_0}T\left(yR\left(k+\tfrac{1}{2}\right)e_q x\right)\\
&= \frac{1}{c_0}T\left(yxR\left(k-\tfrac{1}{2}\right)e_{q+1}\right)\\
&= T\left(\left(k-\frac{1}{2}-\frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i\varepsilon^i g^i}{\varepsilon^i-1}\right)R\left(k-\tfrac{1}{2}\right)e_{q+1}\right),
\end{aligned}
$$

and using Lemma 3.3 and the commutativity of e_q and k once again, we obtain

$$
T\left(\left(k+\tfrac{1}{2}-\alpha_q\right)R\left(k+\tfrac{1}{2}\right)e_q\right) = T\left(\left(k-\tfrac{1}{2}-\alpha_q\right)R\left(k-\tfrac{1}{2}\right)e_{q+1}\right),
$$

as desired.

(8) This follows from conjugation by h.

We now prove the converse. Suppose T is a linear map from H_c to $\mathbb{C}$ satisfying the listed conditions. We wish to show that for all a in H_c, we have $T([x,a]) = T([y,a]) = T([g,a]) = T([h,a]) = 0$. The equality $T([g,a]) = T([h,a]) = 0$ follows from Condition (1), so it remains to verify the result for commutation by x and y. Also, the statements that $T([y,a]) = 0$ for all a follows from the statement that $T([x,a]) = 0$ for all a by conjugation by h and Condition (1). It therefore suffices to show that $T([x,a]) = 0$ when a is equal to a basis element of the form $x^iR(k)e_q$, $y^iR(k)e_q$, $x^iR(k)e_qh$, or $y^iR(k)e_qh$. We now check all possible cases.

- Let $a = x^iR(k)e_q$ with i nonnegative or $a = y^iR(k)e_q$ with $i > 1$. Then $T(xa) = 0 = T(ax)$ by Condition (2).

- Let $a = yR(k)e_q$. Then, using Lemma 3.3, we obtain

$$
\begin{aligned}
T(xa) &= T\left(xyR(k)e_q\right)\\
&= c_0T\left(\left(k+\frac{1}{2}-\frac{1}{c_0}\sum_{i=1}^{n-1}\frac{c_i g^i}{\varepsilon^i-1}\right)R(k)e_q\right)\\
&= c_0T\left(\left(k+\tfrac{1}{2}-\alpha_q\right)R(k)e_q\right).
\end{aligned}
$$

Using Condition (7) and Lemma 3.3, we obtain

$$\begin{aligned}
T(xa) &= c_0 T\left(\left(k - \tfrac{1}{2} - \alpha_q\right) R(k-1)e_{q+1}\right) \\
&= c_0 T\left(\left(k - \frac{1}{2} - \frac{1}{c_0}\sum_{i=1}^{n-1} \frac{c_i \varepsilon^i g^i}{\varepsilon^i - 1}\right) R(k-1)e_{q+1}\right) \\
&= T\left(yxR(k-1)e_{q+1}\right) \\
&= T\left(yR(k)e_q x\right) \\
&= T(ax),
\end{aligned}$$

as desired.

- Let $a = x^i R(k)e_q h$ or $a = y^i R(k)e_q h$, with i even. Then by Condition (3), $T(xa) = 0 = T(ax)$.

- Let $a = x^{2i-1}R(k)e_q h$, and let $S(X)$ be the polynomial such that $S(k)e_{q-1} = xye_{q-1}$. Then by Condition (4),

$$\begin{aligned}
T(xa) &= T\left(x^{2i}R(k)e_q h\right) \\
&= T\left(x^i y^i R(k+i)e_{q-i}h\right) \\
&= T\left(x^{i-1}(xy)y^{i-1}R(k+i)e_{q-i}h\right) \\
&= T\left(x^{i-1}(xy)y^{i-1}e_{q-i}R(k+i)h\right) \\
&= T\left(x^{i-1}(xy)e_{q-1}y^{i-1}R(k+i)h\right) \\
&= T\left(x^{i-1}S(k)e_{q-1}y^{i-1}R(k+i)h\right) \\
&= T\left(x^{i-1}y^{i-1}S(k+i-1)R(k+i)e_{q-i}h\right) \\
&= T\left(x^{2i-2}S(k)R(k+1)e_{q-1}h\right) \\
&= T\left(x^{2i-2}(xy)R(k+1)e_{q-1}h\right) \\
&= T\left(x^{2i-1}R(k)e_q yh\right) \\
&= T(x^{2i-1}R(k)e_q hx) \\
&= T(ax),
\end{aligned}$$

as needed.

- Let $a = y^{2i+1}R(k)e_q h$, and let $S(X)$ be the polynomial such that $S(k)e_{q+2i} = xye_{q+2i}$. Then by Condition (5),

$$\begin{aligned}
T(xa) &= T\left(xy^{2i+1}R(k)e_q h\right) \\
&= T\left(xye_{q+2i}y^{2i}R(k)h\right) \\
&= T\left(S(k)e_{q+2i}y^{2i}R(k)h\right) \\
&= T\left(y^{2i}S(k+2i)R(k)e_q h\right) \\
&= (-1)^i T\left(y^i x^i S(k+i)R(k-i)e_{q+i}h\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned} T(ax) &= T\left(y^{2i+1}R(k)e_q yh\right) \\ &= T\left(y^{2i+2}R(k+1)e_{q-1}h\right) \\ &= (-1)^{i+1}T\left(y^{i+1}x^{i+1}R(k-i)e_{q+i}h\right). \end{aligned} \tag{1}$$

So by Condition (6), both $T(xa)$ and $T(ax)$ are proportional to $T(e_{q+i}h)$, which, by Lemma 4.1, is nonzero only if $m \mid q+i$. We may therefore assume that $m \mid q+i$, because otherwise, $T(xa) = 0 = T(ax)$. It follows that $n \mid 2q+2i$, so $\beta_{q+2i} = \beta_{-q}$. Therefore, by Lemma 3.4,

$$\begin{aligned} T(xa) &= (-1)^i T\left(y^i x^i S(k+i)R(k-i)e_{q+i}h\right) \\ &= T\left(\left(\prod_{j=0}^{i-1} c_0(-k+j) + \beta_{-q-i+j}\right)\left(c_0(k+i) + \beta_{q+2i}\right)R(k-i)e_{q+i}h\right) \\ &= T\left(\left(\prod_{j=0}^{i-1} c_0(-k+j) + \beta_{-q-i+j}\right)\left(c_0(k+i) + \beta_{-q}\right)R(k-i)e_{q+i}h\right). \end{aligned}$$

By Condition (6), this expression depends only on its value for $k = 0$. Therefore, simplifying the expression for $T(xa)$ and using Equation (1), we obtain

$$\begin{aligned} T(xa) &= T\left(\left(\prod_{j=0}^{i-1} c_0(-k+j) + \beta_{-q-i+j}\right)\left(c_0(-k+i) + \beta_{-q}\right)R(k-i)e_{q+i}h\right) \\ &= T\left(\left(\prod_{j=0}^{i} c_0(-k+j) + \beta_{-q-i+j}\right)R(k-i)e_{q+i}h\right) \\ &= (-1)^{i+1}T\left(y^{i+1}x^{i+1}R(k-i)e_{q+i}h\right) \\ &= T(ax), \end{aligned}$$

as needed.

This resolves all cases, completing the proof. □

Theorem 4.2 reduces traces on H_c to traces on a subalgebra and two additional parameters, as shown by the following corollary.

Corollary 4.3. *A trace is uniquely determined by its values on elements of the form $R(k)e_q$, with R a polynomial, and on the conjugacy classes $g^{2i}h$ and $g^{2i+1}h$ of $\mathbb{D}_n$. Also, any choice of linear function T on the subalgebra of H_c spanned by elements of the form $R(k)e_q$ which satisfies Conditions* (7) *and* (8) *of Theorem 4.2 and of values $T(g^{2i}h)$ and $T(g^{2i+1}h)$ defines a unique trace on H_c through Conditions* (1)*–*(6) *of Theorem 4.2.*

Proof. By Theorem 4.2, a trace is uniquely determined by its values on terms $R(k)e_q$ and on the conjugacy classes $g^{2i}h$ and $g^{2i+1}h$. We show that any choice of linear function on terms $R(k)e_q$ which satisfies Conditions (7) and (8) of Theorem 4.2 and of values $T(g^{2i}h)$ and $T(g^{2i+1}h)$ gives a valid trace. Indeed, values of T on basis elements are determined by Conditions (2)–(6) of Theorem 4.2, so all that is left to check is that Condition (1) of Theorem 4.2 does not lead to a contradiction. Using the fact that $ghg^{-1} = g^2h$ and $g^ie_q = \varepsilon^{-iq}e_q$, we see that conjugation by g preserves the relations among basis elements given in Conditions (2)–(6) of Theorem 4.2 and preserves conjugacy classes of $\mathbb{D}_n$. Conjugation by h also preserves relationship among basis elements, so the trace is indeed well-defined. □

We now compute the dimension of the space of traces on H_c.

Proposition 4.4. *The dimension of the space of traces on H_c is $\frac{n}{2}+2$.*

Proof. By Corollary 4.3, it suffices to show that the dimension of the space of linear maps satisfying Conditions (7) and (8) of Theorem 4.2 is $\frac{n}{2}$. Let $V = \mathbb{C}[X]^{\oplus n}$, and define the linear map $\varphi\colon V \to V$ by

$$\begin{aligned}&\varphi(R_0(X), \dots, R_{n-1}(X))\\ &= (R_0(X+\tfrac{1}{2}) - R_{n-1}(X-\tfrac{1}{2}),\ \dots,\ R_{n-1}(X+\tfrac{1}{2}) - R_{n-2}(X-\tfrac{1}{2})),\end{aligned}$$

where the ith component is equal to $R_i(X+\frac{1}{2}) - R_{i-1}(X-\frac{1}{2})$. Define the ideal I of V by

$$I = \{((X-\alpha_0)R_0(X),\ \dots,\ (X-\alpha_{n-1})R_{n-1}(X)) : R_i(X) \in \mathbb{C}[X]\}$$

and the subspace W of V as the set of elements $(R_0(X), \dots, R_{n-1}(X))$ for which $R_i(X) = -R_{-i}(-X)$. The space of linear maps satisfying Conditions (7) and (8) of Theorem 4.2 is isomorphic to the vector space $(V/\langle\varphi(I), W\rangle)^*$, where $\langle\varphi(I), W)\rangle$ denotes the linear span of $\varphi(I)$ and W. We therefore wish to show that the dimension of $V/\langle\varphi(I), W\rangle$ is $\frac{n}{2}$.

First, note that the map φ is surjective. Indeed, to solve the equation $\varphi((R_0, \dots, R_{n-1})) = (P_0, \dots, P_{n-1})$, let R_{n-1} be a solution to the equation

$$R_{n-1}(X+n-\tfrac{1}{2}) - R_{n-1}(X-\tfrac{1}{2}) = \sum_{i=0}^{n-1} P_i(X+i).$$

Then $R_0, \dots, R_{n-2}$ are uniquely defined by the first $n-1$ components of the equation, and the resulting solution is valid because the sum $\sum_{i=0}^{n-1}(R_i(X+i+\frac{1}{2}) - R_{i-1}(X+i-\frac{1}{2}))$ is equal to $R_{n-1}(X+n-\frac{1}{2}) - R_{n-1}(X-\frac{1}{2})$, which equals $\sum_{i=0}^{n-1} P_i(X+i)$, as needed. We also note that the kernel of φ is $\{(z, \dots, z) : z \in \mathbb{C}\}$.

Letting

$$U = \{(R_0(X),\ \dots,\ R_{n-1}(X)) : R_i(X) = R_{-i-1}(-X)\},$$

we claim that the preimage of W under φ is U. Indeed, $\varphi(U) \subseteq W$ because for any element $(R_0(X), \dots, R_{n-1}(X)) \in U$, the ith component of $\varphi(R_0(X), \dots, R_{n-1}(X))$ is

$$R_i(X+\tfrac{1}{2}) - R_{i-1}(X-\tfrac{1}{2}) = R_{-i-1}(-X-\tfrac{1}{2}) - R_{-i}(-X+\tfrac{1}{2}),$$

which is equal to the negative of the $(n-i)$th component. Conversely, for any element $(R_0(X), \ldots, R_{n-1}(X))$ mapped to W by φ, the element of V with ith component equal to $\frac{R_i(X)+R_{-i-1}(-X)}{2}$ belongs to U and is also mapped to W by φ. Therefore,

$$V/\langle \varphi(I), W\rangle \cong V/\langle I, U\rangle.$$

Let $\psi : V \to V/I \cong \mathbb{C}^n$ be the quotient map given by mapping $(R_0(X), \ldots, R_{n-1}(X))$ to $(R_0(\alpha_0), \ldots, R_{n-1}(\alpha_{n-1}))$. Because $\alpha_i = -\alpha_{-i-1}$, $\psi(U)$ is the set S of vectors $(v_0, \ldots, v_{n-1})$ such that $v_i = v_{-i-1}$ for all i. Therefore,

$$V/\langle I, U\rangle \cong \mathbb{C}^n/S,$$

the dimension of which is $\frac{n}{2}$, as desired. □

5 Analytic Formula for Traces

Henceforth, we assume that $|\operatorname{Re}(\alpha_q)| < \frac{1}{2}$ for all q. Define the polynomial

$$\mathbf{P}(X) = \prod_{q=0}^{n-1} \left(X - \exp\left(\tfrac{2\pi i}{n}\left(\alpha_q - q - \tfrac{1}{2}\right)\right)\right),$$

so that $\mathbf{P}\left(\exp\left(\frac{2\pi i}{n} z\right)\right) = 0$ when $z \equiv \alpha_q - q - \frac{1}{2} \pmod{n}$. We observe that the only root of $\mathbf{P}\left(\exp\left(\frac{2\pi i}{n}(z - q - \frac{1}{2})\right)\right)$ with $\operatorname{Re}(z) < \frac{1}{2}$ is at $z = \alpha_q$. We also need the following lemma about $\mathbf{P}(X)$.

Lemma 5.1. *The polynomial $\boldsymbol{P}(X)$ satisfies $\boldsymbol{P}(X) = X^n \boldsymbol{P}(X^{-1})$.*

Proof. For each root $\exp\left(\frac{2\pi i}{n}(\alpha_q - q - \frac{1}{2})\right)$ of $\mathbf{P}(X)$, the complex number

$$\begin{aligned}\exp\left(-\tfrac{2\pi i}{n}\left(\alpha_q - q - \tfrac{1}{2}\right)\right) &= \exp\left(\tfrac{2\pi i}{n}\left(\alpha_{-q-1} + q + \tfrac{1}{2}\right)\right) \\ &= \exp\left(\tfrac{2\pi i}{n}\left(\alpha_{-q-1} - (-q-1) - \tfrac{1}{2}\right)\right)\end{aligned}$$

is also a root of $\mathbf{P}(X)$, so the reciprocal of each root of $\mathbf{P}(X)$ is also a root of $\mathbf{P}(X)$. The claim follows. □

We now give an analytic description of traces.

Theorem 5.2. *Define*

$$\begin{aligned} w_0(z) &= \exp\left(\tfrac{2\pi i}{n} z\right) \cdot \frac{G(\exp\left(\frac{2\pi i}{n} z\right))}{\boldsymbol{P}(\exp\left(\frac{2\pi i}{n} z\right))}, \\ w_q(z) &= w_0(z - q), \end{aligned}$$

where G is a polynomial of degree at most $n-2$ such that $G(X) = X^{n-2} G(X^{-1})$. All traces $T\colon H_c \to \mathbb{C}$ are given by

$$T(R(k)e_q) = \int_{i\mathbb{R}} R(z) w_q(z) |\mathrm{d}z|$$

on $\mathbb{C}[k]e_q$, and extended to H_c using Corollary 4.3 and the values of T on the conjugacy classes $g^{2i}h$ and $g^{2i+1}h$.

Proof. The weight functions w_q satisfy the following properties:

(1) $w_q(z) = w_{q-1}(z-1)$, and so $w_q(z) = w_q(z-n)$;

(2) $w_q(z)$ decays exponentially when $\mathrm{Im}(z)$ approaches $\pm\infty$;

(3) $(z-\alpha_q)w_q(z-\frac{1}{2})$ is holomorphic when $|\mathrm{Re}(z)| \le \frac{1}{2}$.

Indeed, Property (2) holds because $\mathbf{P}(X)$ is not divisible by X and G has degree $\le n-2$, and Property (3) holds because the only root of $\mathbf{P}(\exp(\frac{2\pi i}{n}(z-\frac{1}{2}-q)))$ for $|\mathrm{Re}(z)| \le \frac{1}{2}$ is at $z=\alpha_q$, and the function $\frac{z-\alpha_q}{\exp(\frac{2\pi i}{n}z)-\exp(\frac{2\pi i}{n}\alpha_q)}$ is holomorphic at $z=\alpha_q$.

We now check Conditions (7) and (8) of Theorem 4.2. First,

$$
\begin{aligned}
&T\left(\left(k+\tfrac{1}{2}-\alpha_q\right)R\left(k+\tfrac{1}{2}\right)e_q\right) - T\left(\left(k-\tfrac{1}{2}-\alpha_q\right)R\left(k-\tfrac{1}{2}\right)e_{q+1}\right)\\
&= \int_{i\mathbb{R}} \left(z+\tfrac{1}{2}-\alpha_q\right)R(z+\tfrac{1}{2})w_q(z)|\mathrm{d}z| - \int_{i\mathbb{R}} \left(z-\tfrac{1}{2}-\alpha_q\right)R\left(z-\tfrac{1}{2}\right)w_{q+1}(z)|\mathrm{d}z|\\
&= \int_{\frac{1}{2}+i\mathbb{R}} (z-\alpha_q)R(z)w_q\left(z-\tfrac{1}{2}\right)|\mathrm{d}z| - \int_{-\frac{1}{2}+i\mathbb{R}} (z-\alpha_q)R(z)w_{q+1}\left(z+\tfrac{1}{2}\right)|\mathrm{d}z|\\
&= \int_{\frac{1}{2}+i\mathbb{R}} (z-\alpha_q)R(z)w_q\left(z-\tfrac{1}{2}\right)|\mathrm{d}z| - \int_{-\frac{1}{2}+i\mathbb{R}} (z-\alpha_q)R(z)w_q\left(z-\tfrac{1}{2}\right)|\mathrm{d}z|\\
&= \frac{1}{i}\int_{\partial\left(\left[-\frac{1}{2},\frac{1}{2}\right]\times\mathbb{R}\right)} (z-\alpha_q)R(z)w_q\left(z-\tfrac{1}{2}\right)\mathrm{d}z,
\end{aligned}
$$

which is equal to 0 because $(z-\alpha_q)R(z)w_q(z-\frac{1}{2})$ is holomorphic when $|\mathrm{Re}(z)| \le \frac{1}{2}$ by Property (3) of w_q listed at the beginning of the proof, and hence has no poles in $[-\frac{1}{2},\frac{1}{2}]\times\mathbb{R}$. Thus Condition (7) of Theorem 4.2 holds.

To show that Condition (8) holds, we first show that w_0 is even. Indeed,

$$
\begin{aligned}
w_0(z) &= \exp\left(\tfrac{2\pi i}{n}z\right)\cdot\frac{G\left(\exp\left(\frac{2\pi i}{n}z\right)\right)}{\mathbf{P}\left(\exp\left(\frac{2\pi i}{n}z\right)\right)}\\
&= \exp\left(\tfrac{2\pi i}{n}z\right)\cdot\frac{\exp\left((n-2)\cdot\frac{2\pi i}{n}z\right)G\left(\exp\left(-\frac{2\pi i}{n}z\right)\right)}{\exp\left(n\cdot\frac{2\pi i}{n}z\right)\mathbf{P}\left(\exp\left(-\frac{2\pi i}{n}z\right)\right)}\\
&= \exp\left(-\tfrac{2\pi i}{n}z\right)\cdot\frac{G\left(\exp\left(-\frac{2\pi i}{n}z\right)\right)}{\mathbf{P}\left(\exp\left(-\frac{2\pi i}{n}z\right)\right)}\\
&= w_0(-z).
\end{aligned}
$$

It follows that

$$\begin{aligned}
T\left(R(k)e_q\right) &= \int_{i\mathbb{R}} R(z)w_q(z)|\mathrm{d}z| \\
&= \int_{i\mathbb{R}} R(z)w_0(z-q)|\mathrm{d}z| \\
&= \int_{i\mathbb{R}} R(-z)w_0(-z-q)|\mathrm{d}z| \\
&= \int_{i\mathbb{R}} R(-z)w_0(z+q)|\mathrm{d}z| \\
&= \int_{i\mathbb{R}} R(-z)w_{-q}(z)|\mathrm{d}z| \\
&= T\left(R(-k)e_{-q}\right),
\end{aligned}$$

so Condition (8) of Theorem 4.2 holds.

Finally, the dimension of the space of possible polynomials G is $\frac{n}{2}$ and distinct polynomials yield distinct traces, so, taking into account the value of the trace on the conjugacy classes $g^{2i}h$ and $g^{2i+1}h$, the dimension of the space of traces described by the theorem is $\frac{n}{2}+2$, which is equal to the dimension of the space of all traces by Proposition 4.4. □

6 Positive Traces

In this section, we study positive traces on $\mathcal{O}_c$ and prove our main result. Recall that the antilinear automorphism $\rho\colon H_c \to H_c$ is defined by $\rho(x) = y$, $\rho(y) = -x$, $\rho(g) = g$, $\rho(h) = h$. It follows that $\rho(k) = -k$ and $\rho(e_q) = e_{-q}$. We again assume that $|\operatorname{Re}(\alpha_q)| < \frac{1}{2}$ for all q.

We use the following lemma.

Lemma 6.1 ([6, Lemma 4.2])**.** *Let $w(z)$ be a measurable nonnegative function on $\mathbb{R}$ such that $w(z) < c\exp(-b|z|)$ for some positive constants b, c and which is positive almost everywhere.*

(1) *If $H(z)$ is a continuous complex-valued function on $\mathbb{R}$ with finitely many zeros and at most polynomial growth at infinity, then the set $\{H(z)S(z) : S(z) \in \mathbb{C}[z]\}$ is dense in the space $L^p(\mathbb{R}, w)$.*

(2) *The closure of the set $\{S(z)\bar{S}(z) : S(z) \in \mathbb{C}[z]\}$ in $L^p(\mathbb{R}, w)$ is the subset of almost everywhere nonnegative functions.*

We prove a few preliminary results which restrict a positive trace T to be nonzero only on the span of terms of the form $R(k)e_q$, with R a polynomial.

Proposition 6.2. *Any positive trace T on $\mathcal{O}_c$ satisfies $T(e_0h) = 0$.*

Proof. Let $a = R(k)(x^n + y^n)e$ for some even polynomial R. The element $a = ea$ is in $\mathcal{O}_c$, so by hypothesis, we must have $T(a\rho(a)) > 0$. Expanding and using the evenness of R gives

$$\begin{aligned} T(a\rho(a)) &= T\left(R(k)(x^n + y^n)e\bar{R}(k)(x^n + y^n)e\right) \\ &= T\left(R(k)(x^n + y^n)\bar{R}(k)(x^n + y^n)e\right) \\ &= T\left(R(k)(x^n + y^n)\left(x^n\bar{R}(k-n) + y^n\bar{R}(k+n)\right)e\right), \end{aligned}$$

and distributing and using the commutativity relations with k listed in Lemma 3.1, we obtain

$$\begin{aligned} T(a\rho(a)) &= T\left(R(k)\left(x^{2n}\bar{R}(k-n) + x^n y^n \bar{R}(k+n) + y^{2n}\bar{R}(k+n) + y^n x^n \bar{R}(k-n)\right)e\right) \\ &= T\left(x^{2n}R(k-2n)\bar{R}(k-n)e\right) + T\left(x^n y^n R(k)\bar{R}(k+n)e\right) \\ &\quad + T\left(y^{2n}R(k+2n)\bar{R}(k+n)e\right) + T\left(y^n x^n R(k)\bar{R}(k-n)e\right). \end{aligned}$$

Because e_0 is the average of the elements in C_n and e is the average of the elements in $\mathbb{D}_n$, we have $e = \frac{e_0 + e_0 h}{2}$. Expanding the expression for $T(a\rho(a))$ and removing terms equal to 0, we obtain

$$\begin{aligned} T\left(a\rho(a)\right) &= \tfrac{1}{2}T\left(x^{2n}R(k-2n)\bar{R}(k-n)e_0 h\right) + \tfrac{1}{2}T\left(x^n y^n R(k)\bar{R}(k+n)e_0 h\right) \\ &\quad + \tfrac{1}{2}T\left(y^{2n}R(k+2n)\bar{R}(k+n)e_0 h\right) + \tfrac{1}{2}T\left(y^n x^n R(k)\bar{R}(k-n)e_0 h\right) \\ &\quad + \tfrac{1}{2}T\left(x^n y^n R(k)\bar{R}(k+n)e_0\right) + \tfrac{1}{2}T\left(y^n x^n R(k)\bar{R}(k-n)e_0\right). \end{aligned}$$

Using Conditions (4) and (5) of Theorem 4.2 and rearranging terms, we obtain

$$\begin{aligned} T(a\rho(a)) &= \tfrac{1}{2}T\left(x^n y^n R(k-n)\bar{R}(k)e_0 h\right) + \tfrac{1}{2}T\left(x^n y^n R(k)\bar{R}(k+n)e_0 h\right) \\ &\quad + \tfrac{1}{2}T\left(y^n x^n R(k+n)\bar{R}(k)e_0 h\right) + \tfrac{1}{2}T\left(y^n x^n R(k)\bar{R}(k-n)e_0 h\right) \\ &\quad + \tfrac{1}{2}T\left(x^n y^n R(k)\bar{R}(k+n)e_0\right) + \tfrac{1}{2}T\left(y^n x^n R(k)\bar{R}(k-n)e_0\right). \end{aligned}$$

Equating terms containing $y^n x^n$ with terms containing $x^n y^n$ using conjugation by h and combining like terms gives

$$\begin{aligned} T\left(a\rho(a)\right) &= T\left(x^n y^n e_0 h\right)\left(R(n)\bar{R}(0) + R(0)\bar{R}(n)\right) + T\left(x^n y^n R(k)\bar{R}(k+n)e_0\right) \\ &= T(e_0 h)(R(n)\bar{R}(0) + R(0)\bar{R}(n))\prod_{q=0}^{n-1}(c_0 q + \beta_q) \\ &\quad + \int_{i\mathbb{R}} R(z)\bar{R}(z+n)\left(\prod_{q=0}^{n-1}(c_0(z+q) + \beta_q)\right) w_0(z)|\mathrm{d}z|. \end{aligned} \tag{2}$$

The product $\prod_{q=0}^{n-1}(c_0 q + \beta_q)$ in the first term is nonzero by assumption; indeed, by Lemma 3.5, $c_0 q + \beta_q = c_0(q + \frac{1}{2} - \alpha_q)$, which is equal to zero only if $\alpha_q - \frac{1}{2}$ is an integer, which is impossible.

Now assume, for the sake of contradiction, that $T(e_0h)$ is nonzero. Letting $S(z)$ be the polynomial such that $R(z) = S(z - \frac{n}{2})$ and using the evenness of R, we obtain

$$\begin{aligned} T(a\rho(a)) &= T(e_0h)(R(n)\bar{R}(0) + R(0)\bar{R}(n)) \prod_{q=0}^{n-1}(c_0q + \beta_q) \\ &\quad + \int_{i\mathbb{R}} R(-z)\bar{R}(z+n) \left(\prod_{q=0}^{n-1}(c_0(z+q) + \beta_q)\right) w_0(z)|\mathrm{d}z| \\ &= T(e_0h)(S(\tfrac{n}{2})\bar{S}(-\tfrac{n}{2}) + S(-\tfrac{n}{2})\bar{S}(\tfrac{n}{2})) \prod_{q=0}^{n-1}(c_0q + \beta_q) \\ &\quad + \int_{i\mathbb{R}} S(-z-\tfrac{n}{2})\bar{S}(z+\tfrac{n}{2}) \left(\prod_{q=0}^{n-1}(c_0(z+q) + \beta_q)\right) w_0(z)|\mathrm{d}z| \\ &= T(e_0h)(S(\tfrac{n}{2})\bar{S}(-\tfrac{n}{2}) + S(-\tfrac{n}{2})\bar{S}(\tfrac{n}{2})) \prod_{q=0}^{n-1}(c_0q + \beta_q) \\ &\quad + \int_{i\mathbb{R}+\frac{n}{2}} S(-z)\bar{S}(z)w(z)|\mathrm{d}z|, \end{aligned}$$

where w is a function with $|w(z)|$ satisfying the conditions of Lemma 6.1(1). Because $\beta_q = \frac{c_0}{2} - c_0\alpha_q$ by Lemma 3.5, the weight function $w(z)$ is holomorphic between $i\mathbb{R}$ and $i\mathbb{R}+n$. Moving the contour of integration, we obtain

$$T(a\rho(a)) = \psi(S) + \int_{i\mathbb{R}} S(-z)\bar{S}(z)w(z)|\mathrm{d}z|,$$

where $\psi(S)$ is proportional to $S(\frac{n}{2})\bar{S}(-\frac{n}{2}) + S(-\frac{n}{2})\bar{S}(\frac{n}{2})$. By suitable choice of S, we can force $\psi(S)$ to be negative; let S_0 be a polynomial with this property and satisfying that $S_0(z) = S_0(n-z)$, and let $U(z) = (z - \frac{n}{2})^2$, so that $\psi(S_0 - UL) = \psi(S_0)$ for any polynomial L and $U(z) = U(n-z)$. By Lemma 6.1(1), there exists a sequence of polynomials S_i such that US_i tends to S_0 in the space $L^2(i\mathbb{R}, |w(z)| + |w(n-z)|)$. It follows from the symmetry of S_0 and U that $U(z) \cdot \frac{S_i(z)+S_i(n-z)}{2}$ tends to S_0 in the space $L^2(i\mathbb{R}, |w(z)|)$. Therefore, the integral

$$\int_{i\mathbb{R}} \left(S_0(-z) - U(-z) \cdot \frac{S_i(-z) + S_i(n+z)}{2}\right) \left(\bar{S}_0(z) - \bar{U}(z) \cdot \frac{\bar{S}_i(z) + \bar{S}_i(n-z)}{2}\right) w(z)|\mathrm{d}z|$$

approaches 0. On the other hand, $\psi(S_0(z) - U(z) \cdot \frac{S_i(z)+S_i(n-z)}{2}) = \psi(S_0)$ is negative. Therefore, choosing $S(z) = S_0(z) - U(z) \cdot \frac{S_i(z)+S_i(n-z)}{2}$ for large enough i and noting that the resulting polynomial R is even, we obtain $T(a\rho(a)) < 0$, a contradiction. □

We now obtain a similar result for e_mh.

Proposition 6.3. *Any positive trace T on $\mathcal{O}_c$ satisfies $T(e_mh) = 0$.*

Proof. Let $a = (x^n + y^n + S(k))e$ for some even polynomial S. As in Proposition 6.2, $ea = a$, so a is in $\mathcal{O}_c$, and we must have $T(a\rho(a)) = 0$. Expanding and using the evenness of S gives

$$\begin{aligned}T(a\rho(a)) &= T\left((x^n + y^n + S(k))e(x^n + y^n + \bar{S}(k))e\right) \\ &= T\left((x^n + y^n + S(k))(x^n + y^n + \bar{S}(k))e\right) \\ &= T\left((x^{2n} + x^n y^n + y^n x^n + y^{2n})e\right) + T\left(S(k)\bar{S}(k)e\right) \\ &\quad + T\left(x^n(S(k-n) + \bar{S}(k))e\right) + T\left(y^n(S(k+n) + \bar{S}(k))e\right).\end{aligned}$$

We evaluate these terms separately. By Proposition 6.2 and Equation (2) with $R = 1$, we have

$$T\left((x^{2n} + x^n y^n + y^n x^n + y^{2n})e\right) = \int_{i\mathbb{R}} \prod_{q=0}^{n-1} (c_0(z+q) + \beta_q) w_0(z) |\mathrm{d}z|.$$

Splitting into e_0 and $e_0 h$ components and using Proposition 6.2, we find that the second term is simply

$$\begin{aligned}T\left(S(k)\bar{S}(k)e\right) &= \tfrac{1}{2}T\left(S(k)\bar{S}(k)e_0\right) + \tfrac{1}{2}T\left(S(k)\bar{S}(k)e_0 h\right) \\ &= \int_{i\mathbb{R}} S(z)\bar{S}(z) w_0(z) |\mathrm{d}z|.\end{aligned}$$

Again using Proposition 6.2 and applying Conditions (4) and (5) of Theorem 4.2, we find that the sum of the last two terms is equal to

$$\begin{aligned}&\tfrac{1}{2}T\left(x^n(S(k-n) + \bar{S}(k))e_0 h\right) + \tfrac{1}{2}T\left(y^n(S(k+n) + \bar{S}(k))e_0 h\right) \\ &= \tfrac{1}{2}T\left(x^m y^m(S(k-m) + \bar{S}(k+m))e_m h\right) + (-1)^m \tfrac{1}{2}T\left(y^m x^m(S(k+m) + \bar{S}(k-m))e_m h\right),\end{aligned}$$

which, by conjugation by h, is equal to

$$T(x^m y^m(S(k-m) + \bar{S}(k+m))e_m h) = T(e_m h)(S(-m) + \bar{S}(m)) \prod_{q=0}^{m-1} (c_0 q + \beta_{m+q}).$$

Thus

$$\begin{aligned}T(a\rho(a)) = T(e_m h)&(S(-m) + \bar{S}(m)) \prod_{q=0}^{m-1} (c_0 q + \beta_{m+q}) \\ &+ \int_{i\mathbb{R}} \prod_{q=0}^{n-1} (c_0(z+q) + \beta_q) w_0(z) |\mathrm{d}z| \\ &+ \int_{i\mathbb{R}} S(-z)\bar{S}(z) w_0(z) |\mathrm{d}z|.\end{aligned}$$

Assume, for the sake of contradiction, that $T(e_m h)$ is nonzero. As in the proof of Proposition 6.2, the product $\prod_{q=0}^{m-1}(c_0 q + \beta_{m+q})$ is nonzero by Lemma 3.5 and the condition on the α_q, so for a suitable choice S_0 of even S, the expression

$$T(e_m h)(S(-m) + \bar{S}(m)) \prod_{q=0}^{m-1}(c_0 q + \beta_{m+q}) + \int_{i\mathbb{R}} \prod_{q=0}^{n-1}(c_0(z+q) + \beta_q) w_0(z)|\mathrm{d}z|$$

can be made negative. Taking $U(z) = (z-m)(z+m)$, there exists a sequence S_i of polynomials such that US_i approaches S_0 in $L^2(i\mathbb{R}, |w_0(z)|)$ by Lemma 6.1(1), so $U(z) \cdot \frac{S_i(z)+S_i(-z)}{2}$ approaches S_0 in $L^2(i\mathbb{R}, |w_0(z)|)$ because w_0 is even. Setting $S(z) = S_0(z) - U(z) \cdot \frac{S_i(z)+S_i(-z)}{2}$ for large enough i, it follows that $T(a\rho(a)) < 0$, a contradiction. $\square$

We now obtain our main result.

Theorem 6.4. *Let c be generic and contained in $\mathbb{C}[C_n]$, and assume that $|\operatorname{Re}(\alpha_q)| < \frac{1}{2}$ for all q. Any positive trace of a filtered deformation $\mathcal{O}_c^{\mathbb{D}_n}$ of a Kleinian singularity of type D is the restriction of a positive trace on the corresponding filtered deformation $\mathcal{O}_c^{C_n}$ of a Kleinian singularity of type A.*

Proof. We first describe the correspondence between $\mathcal{O}_c^{\mathbb{D}_n}$ and $\mathcal{O}_c^{C_n}$. Just as the algebra $\mathcal{O}_c^{\mathbb{D}_n}$ is defined as $e(\mathbb{C}[x,y]\#\mathbb{D}_n)e$, a subalgebra of H_c, the algebra $\mathcal{O}_c^{C_n}$ is defined as $e_0(\mathbb{C}[x,y]\#C_n)e_0$, also a subalgebra of H_c. The elements of $\mathcal{O}_c^{\mathbb{D}_n}$ are of the form $R(x,y)e$, with $R(x,y)$ a polynomial expression in x and y invariant under the action of $\mathbb{D}_n$, and the elements of $\mathcal{O}_c^{C_n}$ are of the form $S(x,y)e_0$, with $S(x,y)$ a polynomial expression invariant under the action of C_n. The map $R(x,y)e \mapsto R(x,y)e_0$ for invariant polynomials $R(x,y)$ is an injective algebra homomorphism from $\mathcal{O}_c^{\mathbb{D}_n}$ to $\mathcal{O}_c^{C_n}$, so we may identify $\mathcal{O}_c^{\mathbb{D}_n}$ with a subalgebra of $\mathcal{O}_c^{C_n}$. In particular, $\mathcal{O}_c^{\mathbb{D}_n}$ is mapped to the set of elements of $\mathcal{O}_c^{C_n}$ invariant under the involution given by $ke_0 \mapsto -ke_0$, $x^n e_0 \mapsto y^n e_0$, $y^n e_q \mapsto x^n e_0$.

By this inclusion, any trace on $\mathcal{O}_c^{C_n}$ can be restricted to a trace on $\mathcal{O}_c^{\mathbb{D}_n}$. Observing that the formula in Theorem 5.2 is identical to the analytic formula for type A traces obtained by Etingof, Klyuev, Rains, and Stryker [6, Proposition 3.1] with $z = \frac{k}{n}e_0$ and u and v scalar multiples of $\frac{x^n}{c_0 n}e_0$ and $\frac{y^n}{c_0 n}e_0$, respectively, such that the polynomial P satisfying $uv = P(z+\frac{1}{2})$ obtained from Lemma 3.4 is monic, we see that the resulting restricted trace T can be any trace satisfying Theorem 5.2 and the additional condition that $T(e_q h) = 0$ for all q. This condition follows from Lemma 4.1 and Propositions 6.2 and 6.3, so any positive trace on $\mathcal{O}_c^{\mathbb{D}_n}$ is the restriction of some trace on $\mathcal{O}_c^{C_n}$.

Let T be a positive trace on $\mathcal{O}_c^{\mathbb{D}_n}$, and consider its unique even extension to $\mathcal{O}_c^{C_n}$ given by the even weight function w_0 and the analytic formula of Etingof, Klyuev, Rains, and Stryker [6, Proposition 3.1]. We wish to show that T is positive on $\mathcal{O}_c^{C_n}$. Because T is positive on $\mathcal{O}_c^{\mathbb{D}_n}$, we must have $T(e_0 R(k) e_0 \rho(e_0 R(k) e_0)) > 0$ for any even polynomial R, or equivalently

$$\int_{i\mathbb{R}} R(z)\bar{R}(-z) w_0(z)|\mathrm{d}z| > 0. \tag{3}$$

Also, letting P be the polynomial such that $x^n y^n e_0 = P(k+\frac{n}{2})e_0$, we have

$$\int_{i\mathbb{R}} R(z)\bar{R}(-z-n)P(z+\tfrac{n}{2})w_0(z)|\mathrm{d}z| > 0 \tag{4}$$

for all even polynomials R by Equation (2). By Lemma 6.1(2), Equation (3) is equivalent to $w_0(z)$ being nonnegative on $i\mathbb{R}$ because for a sequence of polynomials S_i tending to some almost everywhere nonnegative even function U in $L^1(\mathbb{R}, |w_0(z)|)$, the sequence of polynomials $\frac{S_i(z)+S_i(-z)}{2}$ tends to U in $L^1(\mathbb{R}, |w_0(z)|)$ because w_0 is even.

Also, letting S be the polynomial such that $R(z) = S(z+\frac{n}{2})$ and noting that $P(z+\frac{n}{2})w_0(z)$ is holomorphic between $i\mathbb{R}$ and $\frac{n}{2}+i\mathbb{R}$, Equation (4) is equivalent to the statement that

$$\int_{\frac{n}{2}+i\mathbb{R}} S(z)\bar{S}(-z)P(z)w_0(z-\tfrac{n}{2})|\mathrm{d}z| > 0$$

for all polynomials S such that $S(z) = S(n-z)$. Again using Lemma 6.1(2), it follows that $P(z)w_0(z-\frac{n}{2})$ is nonnegative on $i\mathbb{R}$.

The positivity conditions obtained on P and w_0 are equivalent to those obtained in [6, Proposition 4.4] for type A, so the trace T is positive on $\mathcal{O}_c^{\mathbb{D}_n}$ if and only if it is positive on $\mathcal{O}_c^{C_n}$. Thus T is the restriction of a positive trace on $\mathcal{O}_c^{C_n}$, as desired. □

7 Conclusion

In this paper, we study positive traces on filtered deformations of Kleinian singularities of type D, which are significant in algebra and theoretical physics. Under the assumption that the structure of the filtered deformation of type D is, in some sense, compatible with the structure of deformations of type A Kleinian singularities, we obtain classification results for traces on deformations of Kleinian singularities of type D analogous to those obtained in [6] for type A. Most importantly, we show that, under certain assumptions, all positive traces of filtered deformations of Kleinian singularities of type D are restrictions of positive traces of filtered deformations of Kleinian singularities of type A. Future research should aim to obtain classification results for positive traces of type D in the general case. One possibility for such research is investigating deformations of Kleinian singularities with $n = 4$, when the group defining the Kleinian singularity is isomorphic to the group of quaternion units and has a high level of symmetry.

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References

[1] William Crawley-Boevey and Martin P. Holland. Noncommutative deformations of Kleinian singularities. *Duke Mathematical Journal*, 92(3):605–635, 1998. doi:10.1215/S0012-7094-98-09218-3.

[2] Patrick Du Val. On isolated singularities of surfaces which do not affect the conditions of adjunction (part I.). *Mathematical Proceedings of the Cambridge Philosophical Society*, 30(4):453–459, 1934. doi:10.1017/S030500410001269X.

[3] Patrick Du Val. On isolated singularities of surfaces which do not affect the conditions of adjunction (part II.). *Mathematical Proceedings of the Cambridge Philosophical Society*, 30(4):460–465, 1934. doi:10.1017/S0305004100012706.

[4] Patrick Du Val. On isolated singularities of surfaces which do not affect the conditions of adjunction (part III.). *Mathematical Proceedings of the Cambridge Philosophical Society*, 30(4):483–491, 1934. doi:10.1017/S030500410001272X.

[5] Pavel Etingof and Douglas Stryker. Short star-products for filtered quantizations, I. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 16:014, 2020.

[6] Pavel Etingof, Daniil Klyuev, Eric Rains, and Douglas Stryker. Twisted traces and positive forms on quantized Kleinian singularities of type A. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 17:029, 2021.

[7] Ivan Losev. Deformations of symplectic singularities and orbit method for semisimple Lie algebras. *Selecta Mathematica*, 28(2):30, 2022.