The Classification and the Hilbert Polynomials of the Coloring of Quandles with size 6

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Abstract

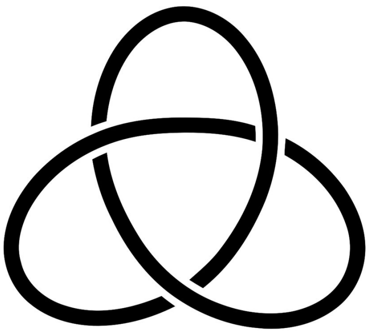
A *quandle* is an algebraic object that is used to define the colorings of knots because quandles axioms ensures that the number of coloring remain invariant. Schlank and Davis showed that the average number of colorings is asymptotically a polynomial, which is the *Hilbert Polynomial*. They also demonstrated the behavior of quandles of up to size 4 and computed the Hilbert Polynomial. We further examine the computations of quandles of size 6 by generalizing constructions of quandles of size 6 except for few, and use the decompositions to compute the Hilbert Polynomial.

Summary

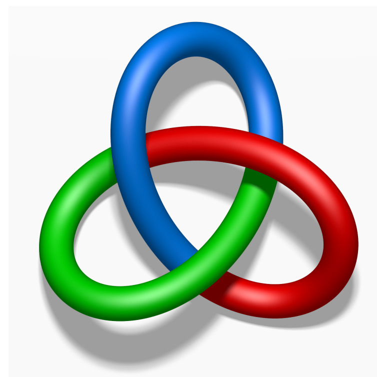
For each knot's arc, we can assign an element from a set. We call this coloring. We may color the knots with multiple different settings. When the knot's crossings are colored, the colorings of arcs need to follow the rule of the set. We call the set with a rule a quandle. Schlank and Davis showed that the average number of coloring can be represented by a polynomial. Based on their finding, we examine further into quandles of size 6. Furthermore, we use new methods to decompose a quandle into smaller quandles. Finally, for all the quandles with 6 elements, we identify the decomposition or show the exotic nature, and compute the polynomial for quandles of size 6.

1 Introduction

A *quandle* is an algebraic object, first introduced by Joyce [1], that is useful for defining *colorings* of oriented link diagrams. The quandle axioms ensure that the total number of colorings of an oriented link diagram by a quandle Q remain invariant under Reidemeister moves, so that we may speak of colorings by Q as an oriented link invariant $\mathrm{Col}_Q(-)$. An subclass of quandles called *involutive quandles*, first defined by Kei [2], define an invariant of unoriented links.



(a) Link Diagram of Trefoil Knot [3]



(b) Colored Trefoil Knot [4]

Figure 1: Coloring of Trefoil Knot

Recently, Schlank and Davis [5] have studied the "average" number of colorings $c_Q(n)$ of the closure of a random braid $\sigma \in B_n$ on n strands (where the average is taken with respect to the Haar measure on the profinite completion $\hat{B}_n$). In particular, they show that $c_Q(n)$ is asymptotically a polynomial in n, which they call the *Hilbert Polynomial* associated to Q. Schlank and Davis show how the Hilbert Polynomial behaves under certain Quandle operations, and compute the Hilbert Polynomial for small quandles of size ≤ 4.

In this work, we take a first step in extending the computations of Schlank and Davis by examining the Hilbert polynomials of involutive quandles of size 6. Our work is helped by the work of Vojtěchovský and Yang [6] who have calculated the total number of quandles of size up to 13 up to isomorphism. However, it is not immediately clear from their computations which larger quandles can be decomposed as to be amenable to the techniques of [5] for computing Hilbert polynomials. We are able generalize the constructions of [5] to express all size 6 involutive quandles except for few in terms of smaller quandles, and in most cases use such decompositions to compute the Hilbert polynomial.

2 Preliminaries

Definition 2.1. An *quandle* Q is a set with a binary operation $(x, y) \mapsto {}^x y$ satisfying

1. ${}^x x = x$
2. ${}^x(-)$ is an bijective map

3. ${}^{x}({}^{y}z) = {}^{({}^{x}y)}({}^{x}z)$.

The quandle Q is *involutive* if

$$ {}^{x}({}^{x}y) = y $$

for all $x, y \in Q$. A *morphism* of quandles is a function $f : Q_1 \to Q_2$ such that for all $x, y \in Q_1$,.

$$ f({}^{x}y) =^{f(x)} f(y). $$

Example 2.1. The *trivial quandle* T_n on n elements is the set $\{1, \ldots, n\}$ with the following binary operator.

$$ \text{For all } x, y \in X,^{x} y = y $$

Example 2.2. The *Dihedral quandle* D_3 has set $\{0, 1, 2\}$ with binary operation ${}^{x}y = 2x - y$ i.e.:

${}^{x}y$		x		
		1	2	3
y	1	1	3	2
	2	3	2	1
	3	2	1	3

The formula ${}^{x}y = 2x - y$ defines a quandle structure on the elements of any abelian group, but the resulting quandle is in general not involutive.

Example 2.3. Let G be a group. Then there is a canonical quandle structure on G given by conjugation ${}^{h}g := hgh^{-1}$ that satisfies all quandle axioms. This extends to a functor $\text{Conj} : \text{Grp} \to \text{Qdl}$.

Construction 2.1. The set Q^n admits an action by the braid group on n strands B_n such that

$$ \sigma_i(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, {}^{x_i}x_{i+1}, x_i, \ldots, x_n) $$

where σ_i is the braid that swaps strands i and $i+1$ by sending strand $i+1$ underneath strand i; see [5, Proposition 2.19].

Theorem 2.1 ([5])**.** *Let $\mathcal{A}_{Q,n} = Q^n/B_n$. Then there exists a polynomial $P_Q(n) \in \mathbb{Q}(n)$ such that $P_Q(n) = |\mathcal{A}_{Q,n}|$ for all $n \gg 0$.*

The size of $\mathcal{A}_{Q,n}$ can be interpreted as the average number of colorings of the closure of a random braid on n strands according to [5, §2]. For finite quandles, a more refined invariant than the Hilbert polynomial is the *generating function*:

Definition 2.2. Let Q be a finite quandle. We define the *generating function* for Q as

$$ \eta_Q(t) = \sum_{t=0}^{\infty} |A_{Q,n}| \cdot t^n \in \mathbb{Z}[\![t]\!] $$

Schlank and Davis show that the Hilbert polynomials of products and disjoint unions of quandles can be computed as follows:

Theorem 2.2. *Let Q and R be finite quandles. The set $Q \times R$ has a natural quandle structure*

$$^{(x,y)}(x', y') = (^x x', ^y y').$$

Proposition 2.3. *Let Q and R be finite quandles. Then for all $n \in \mathbb{N}$ there is an equality:*

$$P_Q(n) \cdot P_R(n) = P_{Q \times R}(n).$$

Theorem 2.4. *Let Q, R be quandles. The disjoint union ,$Q \sqcup R$, has a natural quandle structure given by*

$$(x, y) \in Q \sqcup R :^x y =$$

$^x y$	$x \in Q$	$x \in R$
$y \in Q$	$^x y$	y
$y \in R$	y	$^x y$

Proposition 2.5 ([5])**.** *Let quandles Q and R . Then,*

$$\eta_{Q \sqcup R} = \eta_Q(t) \cdot \eta_R(t).$$

Let Q be a finite quandle and $\psi \in \mathrm{Aut}(Q)$ such that for all $x \in Q$

$$\varphi_\psi(x) = \varphi(x).$$

Proposition 2.6. *Let Q be a quandle and $\psi \in \mathrm{Aut}(Q)$ as above. Then*

$^x y$	$x = *$	$x \in Q$
$y = *$	$*$	$*$
$y \in Q$	$\psi(y)$	$^x y$

is a quandle structure on the set $Q \sqcup_*$. We denote this by $Q \sqcup_{\psi^*}$. The canonical embedding of Q into $Q \sqcup_{\psi^*}$ is a morphism of quandles.

Example 2.4. Let $Q = T_2 = \{a^+, a^-\}$ and let ψ be the non-trivial permutation. Then $J := T_2 \sqcup_\psi$ has structure given by

$^x y$		x		
		a^+	a^-	b
y	a^+	a^+	a^+	a^-
	a^-	a^-	a^-	a^+
	b	b	b	b

This J was first introduced by Joyce [1, §6] as an example of a quandle that does not embed into $\mathrm{Conj}(G)$ for any group G.

3 Quandles of size 6

From computations of Yang and Vojtěchovský [6], we have that there are 73 quandles of size 6, 41 of which are involutive, as summarized in Appendix A. The notation describes the permutations of Q under the action of each of its elements; note that since Q is involutive no such permutation can have a cycle of length more than 2.

Appendix A shows which quandles are able to be written in terms of the ingredients in Section 2. In order to describe more quandles, we introduce the following constructions:

Definition 3.1. Let $f_1 : Q_1 \to R$ and $f_2 : Q_2 \to R$ be Quandle morphisms. The *fiber product* is

$$Q_1 \times_R Q_2 = \{(q_1, q_2) \in Q_1 \times Q_2 : f_1(q_1) = f_2(q_2)\}.$$

Proposition 3.1. *The fiber product $Q_1 \times_R Q_2$ inherits a natural quandle structure.*

Proof. We check that $Q_1 \times_R Q_2$ is a subquandle of $Q_1 \times Q_2$. If $(x_1, x_2), (y_1, y_2) \in Q_1 \times_R Q_2$, we have

$$f_1({}^{x_1}y_1) = {}^{f_1(x_1)}f_1(y_1) = {}^{f_2(x_2)}f_2(y_2) = f_2({}^{x_2}y_2)$$

as desired. □

Proposition 3.2. *Definition 3.1 is the categorical pullback.*

Proof. This is easily verified in identical fashion to [5, Proposition 5.2]. □

Definition 3.2. Let $\psi_{(-)} : Q \to \mathrm{Aut}(\mathrm{Conj}(R))$ and $\varphi_{(-)} : R \to \mathrm{Aut}(\mathrm{Conj}(Q))$ be morphisms of quandles such that

$$\begin{aligned} {}^{r}r' &= {}^{\psi_q(r)}r' \quad \text{for all } q \in Q, r, r' \in R \\ {}^{q}q' &= {}^{\varphi_r(q)}q' \quad \text{for all } r \in R, q, q' \in Q \end{aligned}$$

The *twisted disjoint union* $Q\ {}_{\varphi}\sqcup_{\psi} R$ is defined by

${}^{x}y$	$x \in Q$	$x \in R$
$y \in Q$	${}^{x}y$	$\varphi_x(y)$
$y \in R$	$\psi_x(y)$	${}^{x}y$

Proposition 3.3. *The twisted disjoint union $Q\ {}_{\varphi}\sqcup_{\psi} R$ admits a natural quandle structure.*

Proof. The only nontrivial axiom to check is the third axiom. For $q, q' \in Q, r \in R$ we have that

$${}^{q}({}^{q'}r) = {}^{q}(\psi_{q'}(r)) = \psi_q(\psi_{q'}(r)) = (\psi_q \psi_{q'} \psi_q^{-1})(\psi_q(r)) = {}^{({}^{q}q')}({}^{q}r).$$

$$\begin{aligned} &{}^{q}({}^{r}q') = {}^{q}(\varphi_r(q')) = {}^{\varphi_r(q)}(\varphi_r(q')) \\ &= {}^{({}^{r}q)}({}^{r}q') = {}^{\varphi_r(q)}(\varphi_r(q')) = \varphi_r({}^{q}q') \\ &= \quad {}^{r}({}^{q}q') = {}^{\psi_q(r)}({}^{q}q') \\ &= {}^{({}^{q}r)}({}^{q}q'). \end{aligned}$$

□

Unfortunately, we do not have a method for computing the hilbert polynomial of general fiber products and disjoint unions.

Example 3.1. The quandle on the set $\{1, 2, 3, 4\}$ whose elements act by the permutations

$$(34), (34), (12), (12)$$

can be written as the fiber product $J \times_{T_2} J$ where the two maps $J \to T_2$ are given by

$$a_+ \mapsto 1, a_- \mapsto 1, b \mapsto 2$$

$$a_+ \mapsto 2, a_- \mapsto 2, b \mapsto 1.$$

It can also be written as the twisted disjoint union $T_2 \,{}_\varphi\!\sqcup_\psi T_2$ where ψ and φ are the constant functions with value the "swap" automorphism $(1, 2)$.

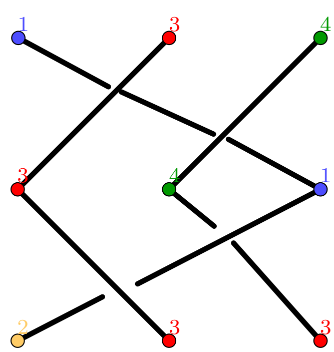
Proposition 3.4. *The Hilbert polynomial of the above quandle is* $P_Q(n) = 4n$.

Proof. We claim that the orbit representatives of Q^n/B_n of the form

$(1, \ldots, 1, 2, \ldots, 2)$ including $(1, \ldots, 1)$ and $(2, \ldots, 2)$
$(3, \ldots, 3, 4, \ldots, 4)$ including $(3, \ldots, 3)$ and $(4, \ldots, 4)$
$(1, \ldots, 1, 3, \ldots, 3)$
$(1, \ldots, 1, 4, 3, \ldots, 3)$ including $(1, \ldots, 1, 4)$.

Given a coloring consisting of all 1's and 2's, we may act by a braid moving all the 1 strands to the left to obtain a representative of the first form above. It is clear there is a distinct orbit for each number of 1's that appear independent orbits since the elements act trivially on each other, so no 1 can be changed to a 2 and vice versa. The same reasoning applies to orbits of the second form above.

Now suppose we are given a coloring with elements of both $\{1, 2\}$ and $\{3, 4\}$. First, let us act by a braid moving all the $\{1, 2\}$ strands to the left and $\{3, 4\}$ strands to the right. Then, we may change all the $\{3, 4\}$ strands to 3's by passing the rightmost 1 or 2 strand all the way to the right under all the $\{3, 4\}$ strands, then back over the 4 strands and under the 3 strands, as shown below:



Similarly, we may do the same in reverse to obtain a representative of the third or fourth form. Finally, we note that a coloring of the third form cannot be mutated to a coloring of the fourth form: in order to keep the $\{1, 2\}$ and $\{3, 4\}$ colors separated, we must act by a braid with an even number of crossings between and $\{1,2\}$ and $\{3,4\}$, which will preserve the parity of the sum of all the elements.

To obtain the Hilbert polynomial, we count the orbits: there are $(n+1)$ orbits each of the first two forms, and $(n-1)$ each of the last two forms, for a total of $4n$. □

In total, using the above constructions we are able to decompose almost every quandle of size 6 as summarized in Appendix A. The exceptions are the quandle whose elements act as

- $(34)(56), (36)(45), (14)(26), (13)(25), (16)(24), (15)(23)$
- $(34)(56), (34)(56), (12), (12), (12), (12)$
- $(56), (34)(56), (56), (56), (), ()$
- $(), (36)(45), (24)(56), (26)(35), (23)(46), (23)(45)$

though they are unknown whether these are fiber products.

4 Discussion

The computations of the Hilbert polynomials of the involutive quandles of size 6 remains unfinished, as does the classification of the quandles above.

A similar examination of involutive or general quandles of size 7 and larger may prove fruitful in identifying patterns in the behavior of the Hilbert polynomials of quandles, in addition of the calculation of the degree performed in [5].

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Appendix

A Quandles of size 6 with classification

Quandle	Decomposition	Hilbert Polynomial
[(), (), (), (), (), ()]	T_6	$\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120}$
[(), (), (), (), (), (3,5)]	$T_5 \sqcup_{\psi^*_{(3,5)}}$	$\frac{(n^2+4n+3)(n^2+4n+4)}{12}$
[(), (), (), (3,5), (), (3,5)]	$J' \sqcup T_2$	$\frac{n^4+14n^3+47n^2+58n+24}{24}$
[(), (3,5), (), (3,5), (), (3,5)]	$J' \sqcup_{\psi^*_{(3,5)}}$	
[(3,5), (3,5), (), (3,5), (), (3,5)]	$(J' \sqcup_{\psi^*_{(3,5)}}) \sqcup_{\psi^*_{(3,5)}}$	
[(), (3,4)(5,6), (), (), (), ()]	$T_5 \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(3,4)(5,6), (3,4)(5,6), (), (), (), ()]	$T_2 \times J$	$2n^2+3n+1$
[(), (), (1,2)(5,6), (1,2)(5,6), (1,2)(3,4), (1,2)(3,4)]	$(J \times_{T_2} J) \sqcup_{\psi^*_{(1,2)}}$	
[(3,4)(5,6), (3,4)(5,6), (1,2)(5,6), (1,2)(5,6), (1,2)(3,4), (1,2)(3,4)]	$(J \times J) \times_{T_3} J$	
[(3,4)(5,6), (3,5)(4,6), (), (), (), ()]	$(T_4 \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(3,5)(4,6)}}$	
[(3,4)(5,6), (3,4)(5,6), (), (), (1,2), (1,2)]	$(J' \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(3,4)(5,6), (3,4)(5,6), (1,2), (1,2), (1,2), (1,2)]	exotic	
[(), (), (5,6), (5,6), (3,4), (3,4)]	$(J \times_{T_2} J) \sqcup T_2$	$\frac{2n^3+6n^2+4n}{3}$
[(), (5,6), (), (), (3,4), (3,4)]	$J' \sqcup_{\psi^*_{(5,6)}}$	
[(), (5,6), (5,6), (5,6), (3,4), (3,4)]	$(J \times_{T_2} J) \sqcup_{\psi^*_{(5,6)}}$	
[(), (3,4)(5,6), (), (), (3,4), (3,4)]	$J' \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(), (3,4)(5,6), (5,6), (5,6), (3,4), (3,4)]	$(J \times_{T_2} J) \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(5,6), (5,6), (), (), (3,4), (3,4)]	$(J' \sqcup_{\psi^*_{(5,6)}}) \sqcup_{\psi^*_{(5,6)}}$	
[(5,6), (5,6), (5,6), (5,6), (3,4), (3,4)]	$((J \times_{T_2} J) \sqcup_{\psi^*_{(5,6)}}) \sqcup_{\psi^*_{(5,6)}}$	
[(5,6), (3,4), (), (), (), ()]	$J \sqcup J$	$2n^3+6n^2+7n+3$
[(5,6), (3,4), (), (), (3,4), (3,4)]	$(J' \sqcup_{\psi^*_{(5,6)}}) \sqcup_{\psi^*_{(3,4)}}$	
[(5,6), (3,4), (5,6), (5,6), (3,4), (3,4)]	$((J \times_{T_2} J) \sqcup_{\psi^*_{(5,6)}}) \sqcup_{\psi^*_{(3,4)}}$	
[(5,6), (3,4)(5,6), (), (), (), ()]	$(T_4 \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(5,6)}}$	
[(5,6), (3,4)(5,6), (), (), (3,4), (3,4)]	$(J' \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(5,6)}}$	
[(5,6), (3,4)(5,6), (5,6), (5,6), (), ()]	exotic	
[(5,6), (3,4)(5,6), (5,6), (5,6), (3,4), (3,4)]	$((J \times_{T_2} J) \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(5,6)}}$	
[(3,4)(5,6), (3,4)(5,6), (), (), (3,4), (3,4)]	$(J' \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(3,4)(5,6), (3,4)(5,6), (5,6), (5,6), (3,4), (3,4)]	$((J \times_{T_2} J) \sqcup_{\psi^*_{(3,4)(5,6)}}) \sqcup_{\psi^*_{(3,4)(5,6)}}$	
[(), (), (), (5,6), (4,6), (4,5)]	$T_3 \sqcup D_3$	$n^3+\frac{3}{2}n^2+\frac{5}{2}n+1$
[(), (), (), (2,3)(5,6), (2,3)(4,6), (2,3)(4,5)]	$T_2 \sqcup_{\psi_{(2,3)}} D_3$	
[(), (), (), (2,3)(5,6), (1,3)(4,6), (1,2)(4,5)]	$T_{3\varphi} \sqcup_{\psi_{D_3}} D_3$	
[(2,3)(5,6), (1,3)(4,6), (1,2)(4,5), (2,3)(5,6), (1,3)(4,6), (1,2)(4,5)]	$T_2 \times D_3$	$6n+6$
[(5,6), (5,6), (1,2), (1,2), (3,4), (3,4)]	$(J_+ \times J_+) \times_\psi T_4$	
[(5,6), (5,6), (1,2), (1,2), (1,2)(3,4), (1,2)(3,4)]	$(J_+ \times J_+) \times_{\psi'} T_4$	
[(5,6), (5,6), (1,2)(5,6), (1,2)(5,6), (1,2)(3,4), (1,2)(3,4)]	$(J_+ \times J_+) \times_{\psi''} T_4$	
[(), (3,6)(4,5), (2,4)(5,6), (2,6)(3,5), (2,3)(4,6), (2,5)(3,4)]	exotic	
[(2,3), (), (), (5,6), (4,6), (4,5)]	$D_3 \sqcup J$	$6n^2-6n+7$
[(2,3), (), (), (2,3)(5,6), (2,3)(4,6), (2,3)(4,5)]	$(D_3 \sqcup_\psi T_2) \sqcup_{\psi^*_{(2,3)}}$	
[(3,4)(5,6), (3,6)(4,5), (1,4)(2,6), (1,3)(2,5), (1,6)(2,4), (1,5)(2,3)]	exotic	
[(2,3), (1,3), (1,2), (5,6), (4,6), (4,5)]	$D_3 \sqcup D_3$	$36n-72$