Bounding The Size of The Modular Pants Graph

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Abstract

The pants graph was first introduced to better understand distinct pants decompositions of hyperbolic surfaces. The action of the mapping class group on the pants graph gives rise to a modular pants graph, which allows for the study of homeomorphism classes of pants decompositions. Using the bijection between homeomorphism classes of pants decompositions and trivalent multigraphs, we provide two upper bounds on the number of vertices in the modular pants graph of genus g: the first is in terms of the number of trivalent multigraphs of genus g with one bridge; the second is a recursive bound in terms of g which makes use of girth.

Summary

In Euclidean geometry, there is only one line through a given point parallel to a given line. There are other types of geometries where this is not true, including one known as hyperbolic geometry. Hyperbolic geometry, just like Euclidean geometry, can be extended to higher dimensions, otherwise known as hyperbolic space. Within this space, there are hyperbolic surfaces. These surfaces can be cut up into smaller surfaces known as pairs of pants, which can be viewed as building blocks for larger surfaces. The number of ways to do so for a given surface is not known; in this paper we provide upper bounds for this quantity.

1 Introduction

Euclid first laid out the foundations of Euclidean geometry, built entirely from five postulates, in *The Elements*. Failed efforts to prove the parallel postulate from the other four led to other geometries where the parallel postulate does not hold, collectively known as non-Euclidean geometry. Hyperbolic geometry is one such geometry. Every hyperbolic surface can be cut up into pairs of pants. For a given surface, this division into pants is not unique, and so the pants graph was introduced in 1980 by Hatcher and Thurston [1] to understand relationships between distinct pants decompositions. More recent work has highlighted the use of graph theory in investigating the structure the pants graph.

A **pair of pants** is a surface of genus 0 with three boundary components, shown in Figure 1. A **pants decomposition** of a surface of genus g is a system of $3g-3$ non-homotopic and non-intersecting simple, closed curves that cuts the surface into $2g-2$ pairs of pants, as described by Putman [2]. An example is in Figure 1. The pants graph is a graph whose vertices represent isotopy classes of pants decompositions and whose edges represent elementary moves that change one pants decomposition into another. One such move is an A-move, as in Hatcher [3], which changes one type of curve into another.

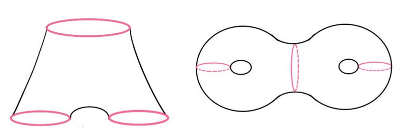

Figure 1: A pair of pants and a pants decomposition of a genus 2 surface

The modular pants graph, described by Putman [2], is a graph where vertices represent homeomorphism classes and where edges represent A-moves only. The number of homeomorphism classes of pants decompositions is not known. The following fact, shown in Putman [2], has led to graph theoretical approaches to this problem: homeomorphism classes of pants decompositions are in bijection with trivalent multigraphs, where a **trivalent multigraph** is any undirected graph where all vertices have degree three and where duplicate edges and loops are allowed. Each individual pair of pants in a pants decomposition corresponds to a vertex in a trivalent multigraph, and a shared boundary between two pairs of pants corresponds to an edge between vertices. An example is provided Figure 2.

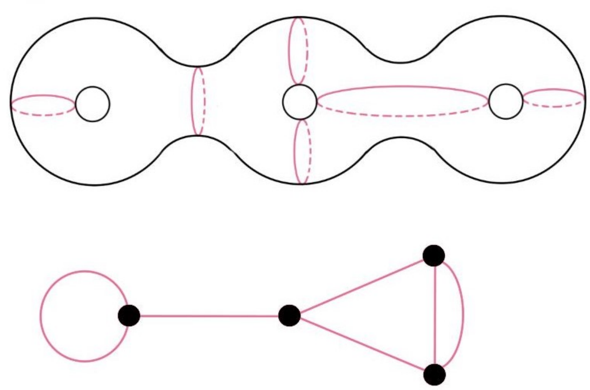

Figure 2: A pants decomposition of a genus 3 surface and its corresponding trivalent multigraph; each pair of pants corresponds to a vertex; a shared boundary corresponds to a shared edge

We can then use trivalent multigraphs to investigate pants decompositions without looking at pants decompositions themselves. In particular, we can describe A-

moves in terms of trivalent multigraphs, as in Sultan [4] and Benvenuti and Piergallini [5]. Sultan provides a bound on the maximum distance in the modular pants graph between a non-separable trivalent multigraph and some separable trivalent multigraph using the graph theoretic concept of girth [4].

The modular pants graph is a combinatorial model of the Weil-Petersson metric on the moduli space of Reimann surfaces, as shown in Brock [6], which is useful for studying hyperbolic space. The Weil-Petersson metric is also of interest to physicists because it appears in calculations in string theory. In addition, the generation of trivalent graphs has long been a problem in both math and computer science, as described by Brinkmann, Goedgebeur and Van Cleemput [7], and cubic graphs are also interesting to chemists for modelling carbon networks.

In Section 2, we go over graph theory and the definition of the modular pants graph. In Section 3 we further explore the structure of the modular pants graph. In Section 4, we provide an upper bound on the number of vertices in the modular pants graph in terms of the number of trivalent multigraphs of genus g with one bridge (Theorem 14). In Section 5, we provide an a recursive upper bound (Theorem 17) for the number of vertices in the modular pants graph in terms of g using girth and A-move's actions on cycles described in Sultan [4].

2 Preliminaries

2.1 Graph Theory

We investigate how A-moves change certain graph theoretical properties of trivalent multigraphs. We are then able to study pants decompositions and the modular pants graph entirely within the realm of graph theory. The following definitions from graph theory are relevant for that purpose.

A **bridge** in a graph G is an edge whose removal from G would result in a disconnected graph. The **girth** of a graph G is the length of the smallest cycle in G. A **full, rooted, binary tree** is a graph which contains no cycles, and where one vertex has degree 2 and all others have degree 3 or 1. A **trivalent multigraph** is an undirected graph whose vertices all have degree 3 and where duplicate edges and loops are allowed.

2.2 Pants Decompositions, A-moves and The Modular Pants Graph

As previously mentioned, the set of homeomorphism classes of pants decompositions is in bijection with the set of trivalent multigraphs. The bijection between pants decompositions and trivalent multigraphs is the following: vertices in trivalent multigraphs correspond to pairs of pants, and two vertices in a trivalent multigraph are connected by an edge if the two pairs of pants they represent share a simple closed curve in the pants decompositions. Thus a trivalent multigraph of genus g

consists of $2g-2$ vertices and $3g-3$ pairs of pants.

An A-move is one of two elementary actions that can be performed on a pants decomposition, the other being an S-move. S-moves do not change the homeomorphism class of the pants decomposition and so are not represented in the modular pants graph. We do not discuss S-moves further in this paper. A-moves can be understood both in terms of pants decompositions and in terms of trivalent multigraphs. An A-move on a pants decomposition is depicted in Figure 3.

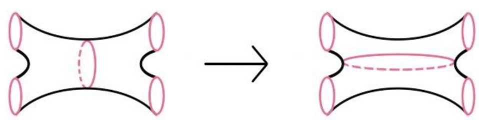

Figure 3: An A-move on part of a pants decomposition

We now define A-moves in terms of trivalent multigraphs, based on definitions used in Sultan [4] and in Benvenuti and Piergallini [5].

Definition 1. An A-move is a mapping from one trivalent multigraph G to the another trivalent multigraph H that is dependent on a chosen edge e in G and a pairing of the edges adjacent to e. Suppose we have an edge e in G which is adjacent to edges a and c at a vertex v_1 and to edges b and d at a vertex v_2, as in Figure 4 An A**-move** with respect to e in G, with pairings (a,b) and (c,d), is a mapping where, if $A_{e(a,b)}(h)$ is the image of an element h (so h is either a vertex or an edge) under the A-move, then:

1. $A_{e(a,b)}(a)$ is adjacent to $A_{e(a,b)}(b)$ and $A_{e(a,b)}(e)$ at $A_{e(a,b)}(v_1)$

2. $A_{e(a,b)}(c)$ is adjacent to $A_{e(a,b)}(d)$ and $A_{e(a,b)}(e)$ at $A_{e(a,b)}(v_2)$

3. all other edge-vertex incidences are preserved as long as they don't involve e, v_1, or v_2

Figure 4 is a depiction of an A-move on a trivalent multigraph. Note that, given an edge e, there are two possible A-moves one can perform on the edge, given by the two different possible pairings of the edges adjacent to e. Unless it is necessary to specifcy the pairing, we will denote an A-move with respect to e as A_e. In addition, note that the labelling is "removed" after the A-move is performed so that we once again have an unlabelled trivalent multigraph. It is possible that an A-move results in the same graph.

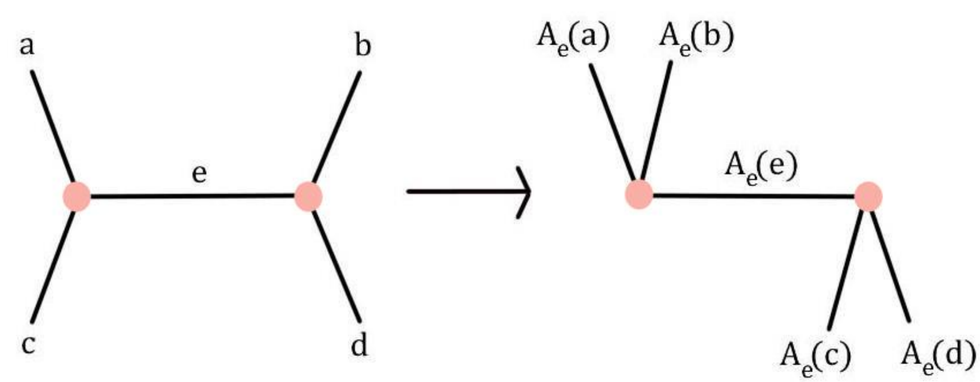

Figure 4: An A-move with respect to e (with pairings (a,b) and (c,d)); vertices are not labelled

With a more concrete concept of an A-move on trivalent multigraphs, we can present a definition of the modular pants graph, from Putman [2].

Definition 2. The **modular pants graph of genus** g, denoted $\mathrm{P}_m(g)$, is the graph where each vertex represents a homeomorphism class of pants decompositions of genus g, and thus a trivalent multigraph, and there exists an edge between vertices if an A-move can be performed on the trivalent multigraph represented by one of the vertices to get to the trivalent multigraph represented by the other vertex.

Trivalent multigraphs, pants decompositions and the vertices of $\mathrm{P}_m(g)$ are then interchangeable in this context. In this paper *graph* is interchangeable with *trivalent multigraph*; *graph* is never used to refer to $\mathrm{P}_m(g)$ or to a graph that is not a trivalent multigraph. Trivalent multigraphs are unlabelled unless we impose a labelling; thus two trivalent multigraphs are considered the same if they are isomorphic.

3 Action of A-moves on Separable Graphs and Separability of Specific Edges

To get a bound for the number of vertices in $\mathrm{P}_m(g)$, we look at the structure of the $\mathrm{P}_m(g)$; in particular, we define the separation number of a trivalent multigraph and the separability of an edge.

Definition 3. A graph G is k**-bridged** if there are exactly k distinct bridges in G; k is the **separation number** of G, denoted $N_s(G)$.

We are not only interested in how separable a graph is at a global scale, but also at how specific edges behave within a graph.

Definition 4. Given a labelling of the edges of G, an edge e in G is **separating** if e is a bridge in G; otherwise e is **non-separating**. Whether or not a given edge is separating is the edge's **separability**.

We now have a crucial lemma about the behavior of A-moves on edge separability.

Lemma 5. *Given a trivalent multigraph G, a labelling of the edges in G and an edge e in G, for all $h \neq e$, $A_e(h)$ has the same separability as h. In other words, an A-move with respect to an edge e preserves the separability of all edges except e.*

Proof. We do separate cases, and the cases are shown in Figure 5.

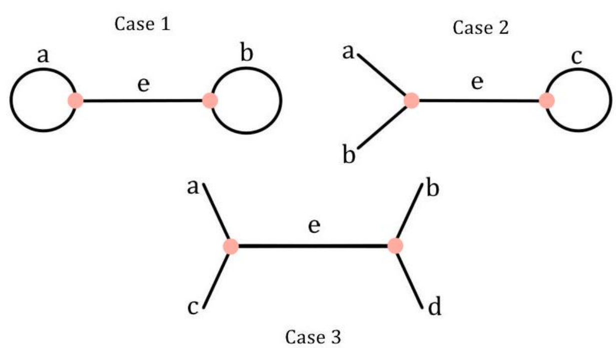

Figure 5: Cases for proof of Lemma 5

Case 1: e is adjacent to two loops, one at each vertex. Then G has only two vertices and three edges. Let one loop be a and the other b. Loops are always non-separating. After performing either of the two possible A-moves on e, we obtain the the only other trivalent graph on 2 vertices, where the vertices share three edges, each of which is non-separating. Thus the separability of the two loops remains the same.

For the next two cases, note that A-move with respect to an edge e changes only those edge adjacencies that occur on a vertex incident to e.

Case 2: e is adjacent to one loop at one vertex, and to two distinct edges at the other. Let the loop be c and let the two distinct edges be a and b. In this case, the two different possible A-moves result in isomorphic graphs. Without loss of generality, we only have to look at b. We have two subcases: either b is separating, or b is non-separating.

If b is non-separating, there exists a path from b to a that does not contain vertices incident to e. This path is unchanged by the A-move, so there exists a path from $A_e(b)$ to $A_e(a)$ that does not contain vertices incident to $A_e(e)$. Then $A_e(b)$ lies on a cycle with $A_e(a)$, and so is non-separating.

If b is separating, there are no other paths from b to a besides the one that passes through a vertex incident to e. Then the only path from $A_e(b)$ to $A_e(a)$ is the one that passes through $A_e(e)$. So $A_e(b)$ is separating.

Case 3: e is adjacent to no loops; e is adjacent to two distinct edges at each vertex, for a total of four distinct edges. Suppose e is adjacent to edges a and c at one vertex, and to edges b and d at the other. Without loss of generality, suppose that we perform an A-move on e with pairings (a, b) and (c, d). We need only look at one of the edges adjacent to e, say a. We have two subcases: either a is a separating, or it is not.

If a is separating, there is exactly one path from a to each of b, c or d, and for each, that path passes through a vertex incident to e. Then in the image of the A-move, there is only one path from $A_e(a)$ to each of $A_e(b)$, $A_e(c)$, and $A_e(d)$, and for each, that is a path that passes through a vertex incident to $A_e(e)$. So $A_e(a)$ is still separating.

If a is non-separating, there is a path from a to one of b, c, or d that contains no vertices incident to e. Then we have a path from $A_e(a)$ to one of $A_e(b)$, $A_e(c)$ or $A_e(d)$ that does not contain either of the vertices incident to $A_e(e)$. Thus $A_e(a)$ is non-separating. $\square$

In particular, this means that, given an edge e, both possible A-moves one can perform on e preserve the separability of all edges except e.

Lemma 6. *Given a k-bridged trivalent multigraph G, with $k \geq 1$, there exists a separating edge e in G so that the image of e under some A-move A_e, $A_e(e)$, is non-separating.*

Proof. First, any separating edge e which is incident to a vertex on a cycle can be made non-separating by an A-move with respect to e. (Because it is separating, the other endpoint of e cannot be part of the same cycle). Suppose e is adjacent to edges a and b at one of its endpoints, so that a and b lie on a cycle with one another. That means there exists a path connecting a and b that touches no vertex incident to e. Then an A-move with respect to e maps to a new graph where $A_e(e)$ is adjacent to $A_e(a)$ at one vertex and to $A_e(b)$ at another. Because the A-move preserves all edge adjacencies except for the adjacencies occurring on a vertex incident to e, $A_e(e)$ is part of a cycle. Thus $A_e(e)$ is no longer separating. Figure 6 depicts this.

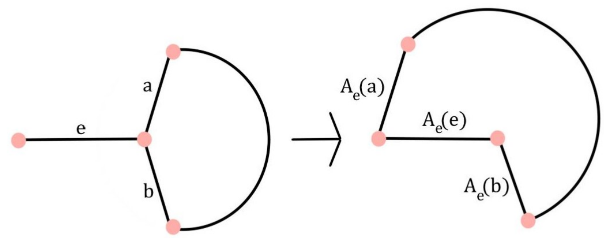

Figure 6: A separating edge e becoming non-separating through an A-move

Now consider some k-bridged graph G with $k \geq 1$. If there is a separating edge incident to a vertex on a cycle, we are done. Otherwise, assume for the sake of contradiction that there are no such separating edges in G. Choose one separating edge e_1. Choose one of the vertices incident to e_1. Because e_1 is not incident to a vertex on a cycle, the two other edges incident to this vertex are each also separating (the only way for them to be non-separating would be if there were a path from one to another that does not pass through a vertex incident to e_1, but this is impossible because neither vertex incident to e_1 lies on a cycle). Choose one of these two new separating edges, say e_2. If e_2 is incident to a vertex on a cycle, we have a contradiction. So then, as before, we choose one of the vertices incident to e_2. Continue on in this way. Since there are a finite number of separating edges, at some point we must choose the same edge again, say e_r, so that e_r is the first edge repeated in our sequence of chosen edges. Then e_r is on a cycle, and so it is non-separating, which is a contradiction. Thus there must exist a separating edge incident to a vertex on a cycle if G contains a positive number of separating edges, and we are done. $\square$

Lemma 5 and Lemma 6 then give rise to relationships between trivalent multigraphs of different separation numbers within $\mathrm{P}_m g$.

Proposition 7. *If a genus g trivalent multigraph G is k-bridged, and another genus g trivalent multigraph H is one A-move away from G in $\mathrm{P}_m(g)$, then H is $(k-1)$-, (k)- or $(k+1)$-bridged.*

Proof. By Lemma 5, the only edge which may have a different separability than its image under an A-move A_e is the edge e. Thus if H is one A-move away from G, then $N_s(H)$ differs from $N_s(G)$ by at most 1. □

We have a more specific result about moving to lower separation number.

Proposition 8. *Given any k-bridged trivalent multigraph G with $k \geq 1$ there exists a $(k-1)$-bridged trivalent multigraph H that is adjacent to G in the $\mathrm{P}_m(g)$. In other words, we can perform an A-move on any k-bridged graph to get to a $(k-1)$-bridged graph.*

Proof. Consider a k-bridged trivalent multigraph G. By Lemma 6, we know that we can find a separating edge that can be made non-separable via a single A-move; by Lemma 5, this A-move changes the separability of no other edges except that separating edge. Thus we can get from G to some $(k-1)$-bridged graph in a single A-move. □

Corollary 9. *The distance between a given separable trivalent multigraph and some nonseparable trivalent multigraph in $\mathrm{P}_m(g)$ is exactly the number of bridges in the separable graph.*

Given the ability to move between graphs of different separation number, we find the largest possible separation number of a trivalent multigraph of genus g.

Definition 10. A **trivalent loop-tree** is a graph whose vertices all have degree 3 and where there are no cycles except for loops.

Figure 7 is an example of a trivalent loop-tree. Deleting any separating edge in a trivalent loop-tree results in two disconnected components. Each component is a construction where a loop is attached to each vertex of degree 1 in a full, rooted binary tree. It is known that any full, rooted binary tree with n vertices has $\frac{(n+1)}{2}$ vertices of degree 1.

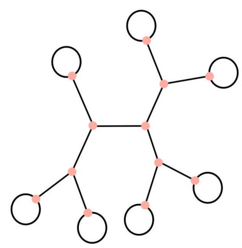

Figure 7: A trivalent loop-tree

Proposition 11. *Trivalent loop-trees of genus g have the maximal number of separating edges, namely $2g-3$.*

Proof. Consider a trivalent loop-tree G of genus g. Choose some separating edge e in G, with endpoints v_1 and v_2. Let K_1 and K_2 be the two disconnected subgraphs of G obtained by deleting e, so that K_1 contains v_1. Suppose K_1 consists of m vertices; then K_2 consists of $2g-2-m$ vertices. The subgraph of K_1 obtained by deleting all of the loops is a full, rooted binary tree with m vertices, so that subgraph has $\frac{m+1}{2}$ leaves. Then K_1 has $\frac{m+1}{2}$ loops. Similarly, K_2 has $\frac{2g-2-m+1}{2}$ loops. The total number of loops in G is then $\frac{m+1}{2} + \frac{2g-2-m+1}{2} = g$, because any loop in G

is either in K_1 or K_2. Besides those loops, all other edges in the entire graph G are separating. Since there are $3g-3$ total edges in G, G is $(2g-3)$-bridged.

Now suppose for the sake of contradiction that we have a graph G which is with separation number $2g-2$. We can delete all the separating edges of G. We have at least $2g-2$ separating edges, so we should have at least $2g-1$ disconnected components. But there are only $2g-2$ vertices in G, a contradiction. Thus the maximal separation number of a trivalent multigraph of genus g is $2g-3$. $\square$

4 Bounding the number of vertices in the modular pants graph using k-bridged graphs

We use moves between k-bridged graphs to get a bound on the size of $\mathrm{P}_m(g)$. For intuition, within $\mathrm{P}_m(g)$, certain graphs must lie in a ball of some radius centered on a graph with some characteristic. Then the number of graphs of one type must be less than the number of vertices that lie within such a ball.

Given $k \geq 0$, the number of k-bridged trivalent multigraphs of genus g is denoted $\mathrm{Sep}_k(g)$. The number of non-separable trivalent multigraphs of genus g, which are the graphs with 0 bridges, is denoted $\mathrm{NSep}(g) = \mathrm{Sep}_0(g)$. The total number of separable graphs is $\mathrm{Sep}(g)$.

Lemma 12. *For $k \geq 1$,*

$$\mathrm{Sep}_k(g) \leq 2(3g-3-(k-1))\,\mathrm{Sep}_{k-1}(g).$$

Proof. By Proposition 8, each k-bridged graph is adjacent to some $(k-1)$-bridged graph. Each $(k-1)$-bridged graph has $3g-3-(k-1)$ non-separating edges. To get to a k-bridged graph, an A-move must be done with respect to one of these non-separating edges. There are two possible A-moves given an edge. So that means there are at most $2(3g-3-(k-1))$ graphs adjacent to a given $(k-1)$-bridged graph. Since every k-bridged graph is adjacent to some $(k-1)$-bridged graph, we have the result. □

We also relate $\mathrm{NSep}(g)$ to $\mathrm{Sep}_1(g)$.

Lemma 13. *For a given genus g,*

$$\mathrm{NSep}(g) \leq (2(3g-3))^{M_g - 1}\,\mathrm{Sep}_1(g),$$

where $M_g = \lfloor 2\log_2(g-1) + 2 \rfloor$.

Proof. From the proof of Lemma 4.3 in Sultan [4], we know that we can reduce the size of the smallest cycle in a graph through successive A-moves on an edge in the smallest cycle until we have a loop and a separating edge (basically a cycle of length 1). The number of A-moves necessary is the girth of the graph minus 1. Because of Lemma 5, this results in a graph with one more bridge than the original graph we started with. Thus the distance between a given k-bridged graph and some $(k+1)$-bridged graph is at most the girth of the k-bridged graph minus 1. If a non-separable graph has girth c, it is at most $c-1$ A-moves away from a 1-bridged graph, because we can do successive A-moves to reduce the size of any cycle in a graph by 1 with each move. The girth of a genus g graph is at most $M_g = \lfloor 2\log_2(g-1)+2\rfloor$, from the proof of Lemma 4.3 in Sultan [4]. All trivalent multigraphs, and thus all 1-bridged trivalent multigraphs, have $3g-3$ edges, upon which 2 A-moves can be performed. Then, given a 1-bridged graph, the number of graphs that are at most $M_g - 1$ A-moves away is at most $(2(3g-3))^{M_g-1}$. This gives the result. □

Then we have a bound for the total number of trivalent multigraphs of genus g, $\mathrm{Tot}(g)$ in terms of $\mathrm{Sep}_1(g)$.

Theorem 14. *For a given genus g,*

$$\mathrm{Tot}(g) \leq \mathrm{Sep}_1(g) \left(\sum_{k=1}^{2g-3} \left(2^{k-1} \frac{(3g-4)!}{(3g-3-k)!} \right) + (2(3g-3))^{M_g-1} \right),$$

where $M_g = \lfloor 2\log_2(g-1) + 2 \rfloor$.

Proof. From Lemma 12, we have

$$\mathrm{Sep}_k(g) \leq 2^{k-1} \frac{(3g-4)!}{(3g-3-k)!} \mathrm{Sep}_1(g).$$

Summing over all k results in a bound for the total number of separable graphs:

$$\mathrm{Sep}(g) \leq \mathrm{Sep}_1(g) \sum_{k=1}^{2g-3} 2^{k-1} \frac{(3g-4)!}{(3g-3-k)!}.$$

Combining this with Lemma 13 gives the proposition. □

From Theorem 14gives the following asymptotic relation. The details of the calculations are left to Appendix B.

$$\mathrm{Tot}(g) \lesssim g^{3g} \mathrm{Sep}_1(g).$$

5 Bounding the number of vertices in the modular pants graph using girth

Sultan[4] showed that it is possible to reduce the girth of a graph by 1 through an A-move on some edge in the cycle of shortest length. This gives rise to another upper bound for the size of $\mathrm{P}_m(g)$. Let $\mathrm{Gir}_c(g)$ be the number of genus g trivalent multigraphs of girth c. Then c can range from 1 to $M_g = \lfloor 2\log_2(g-1)+2 \rfloor$. The maximum girth comes from Sultan [4].

Lemma 15. *For a given genus g and girth c,*

$$\mathrm{Gir}_c(g) \leq 2(3g-3)\,\mathrm{Gir}_{c-1}(g).$$

Proof. We can move from any graph of girth c to a graph of girth $c-1$ in one A-move, as shown in the proof of Lemma 4.3 in Sultan [4]. Thus any graphs of girth c is a distance of 1 away from some graph of girth $c-1$ in $\mathrm{P}_m(g)$. For a given graph of girth $c-1$, there are at most $2(3g-3)$ distinct graphs that are one A-move away. $\square$

We can relate $\mathrm{Gir}_1(g)$ to $\mathrm{Tot}(g-1)$.

Lemma 16. *For a given genus g,*

$$\mathrm{Gir}_1(g) \leq (3(g-1)-3)\,\mathrm{Tot}(g-1).$$

Proof. Every graph of girth 1 and genus g can be viewed as a construction where a vertex with an edge that leads to another vertex with a loop is inserted into some edge of some graph of genus $g-1$. There are $3(g-1)-3$ edges in any trivalent multigraph of genus g. This gives the result. $\square$

We now have an recursive upper bound for $\mathrm{Tot}(g)$.

Theorem 17. *For a given genus g,*

$$\mathrm{Tot}(g) \leq \left(\sum_{c=1}^{M_g} (2(3g-3))^{c-1}\right)(3(g-1)-3)\,\mathrm{Tot}(g-1),$$

where $M_g = \lfloor 2\log_2(g-1)+2 \rfloor$.

Proof. From Lemma 15, we have

$$\mathrm{Gir}_c(g) \leq (2(3g-3))^{c-1}\,\mathrm{Gir}_1(g).$$

Summing over all c results in

$$\mathrm{Tot}(g) \leq \left(\sum_{c=1}^{M_g} (2(3g-3))^{c-1}\right) \mathrm{Gir}_1(g).$$

With Lemma 16, we arrive at the proposition. $\square$

Theorem 17 gives the following asymptotic relation. Details of the calculations are left to Appendix B.

$$\frac{\mathrm{Tot}(g)}{\mathrm{Tot}(g-1)} \lesssim g^{3\log_2(g)+4}.$$

6 Future Work

Rather than looking at separation number and girth separately, it may be productive to consider them simultaneously. In addition, these bounds are extremely generous; investigating symmetries of graphs may result in improvements to the bounds. Moreover, there is still not much known about the structure of the pants graph, so identifying subgraphs and looking at the connectivity of different parts of the pants graph may be interesting.

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A Appendix 1: An upper bound for 1-bridged graphs

We relate $\mathrm{Sep}_1(g)$ to counts of nonseparable graphs of lower genus.

Proposition 18. *For a given genus g,*

$$\mathrm{Sep}_1(g) \leq (3(g-1)-3)\,\mathrm{NSep}(g-1) + \sum_{i=2}^{\lfloor \frac{g}{2} \rfloor} (3i-3)(3(g-i)-3)\,\mathrm{NSep}(i)\,\mathrm{NSep}(g-i).$$

Proof. First, we show that any 1-bridged graph can be viewed as a construction where we take two non-separable trivalent multigraphs, break apart an edge in each by inserting a vertex, and then connecting the two new vertices. Consider some 1-bridged trivalent multigraph G. Its only separating edge, say e, connects two components, say K_1 and K_2 that would be disconnected if e were deleted. Consider just one of the components, say K_1. Because the only separating edge in G is e, there exists a path that doesn't pass through e between any two vertices in K_1. Thus K_1, when e is deleted, still has no bridges. K_1 has only one vertex of degree 2, namely the vertex that was incident to e in G. Let this vertex be v_1; let the two vertices it is adjacent to be $v_{1,1}$ and $v_{1,2}$. Then obtain a new graph K_1' by deleting v and its incident edges, and then drawing an edge between $v_{1,1}$ and $v_{1,2}$. This new K_1' is a non-separable trivalent multigraph. (Any path that passed through v_1 in K_1 either passed through or started/ended on $v_{1,1}$ and $v_{1,2}$, and so in K_1' there are still paths between all possible pairs of vertices.) Thus we can reconstruct G by taking these two non-separable graphs K_1', K_2', replacing a specific edge in each with a vertex of degree 2, and then attaching those two new vertices via another edge.

Thus all 1-bridged graphs of a given genus can be obtained by choosing two non-separable graphs of lower genus, choosing a specific edge to be replaced in each graph, and then attaching the two graphs. To obtain a graph graph of genus g (so $2g-2$ vertices), we would need a non-separable graph of genus i ($2i-2$ vertices) and a non-separable graph of genus $g-i$ ($2(g-i)-2$ vertices); adding in the two additional vertices in our construction results in a total of $2g-2$ vertices. On a graph of genus i, there are $3i-3$ edges. Thus given a specific non-separable graph of genus i and specific non-separable graph of genus $g-i$, there are at most $(3i-3)(3(g-i)-3)$ distinct ways to attach them to each other via the construction described to form a 1-bridged graph of genus g. Note that no two values of i can result in the same constructions, since the two components separated by an edge in the end result would be of different sizes. Thus iterating over all possible values i with $i \leq g-i$ gives the result. Note that there are no trivalent multigraphs of genus 1, which is why $NSep(g-1)$ is on its own. For non-separable graphs of genus $g-1$, we can replace one of its edges with a vertex of degree 2 and its incident edges, and then to that vertex attach a vertex with a loop to obtain a 1-bridged graph of genus g.

$\square$

B Appendix 2: Asymptotic analysis of the bounds

We compute asymptotics for the bounds in Theorem 14 and Theorem 17.

Theorem 14 says

$$\mathrm{Tot}(g) \leq \mathrm{Sep}_1(g)\left(\sum_{k=1}^{2g-3}\left(2^{k-1}\frac{(3g-4)!}{(3g-3-k)!}\right)+(2(3g-3))^{M_g-1}\right),$$

where $M_g = \lfloor 2\log_2(g-1)+2\rfloor$. We compute the asymptotics for this bound in terms of $\mathrm{Sep}_1(g)$. We will split the coefficient of $\mathrm{Sep}_1(g)$ into two parts and calculate the asymptotics of the two parts separately before calculating them for the total expression.

In the sum in the coefficient, the term with the largest growth is the one that corresponds to $k = 2g-3$. We have

$$\begin{aligned}\left(\sum_{k=1}^{2g-3}\left(2^{k-1}\frac{(3g-4)!}{(3g-3-k)!}\right)\right) &\sim 2^{2g-4}\frac{(3g-4)!}{g!}\\ &\sim 2^{2g-4}\frac{\sqrt{2\pi(3g-4)}(\frac{3g-4}{e})^{3g-4}}{\sqrt{2\pi g}(\frac{g}{e})^g}\\ &= (\frac{2}{e})^{2g-4}\sqrt{\frac{3g-4}{g}}\frac{(3g-4)^{3g-4}}{g^g}.\end{aligned}$$

where we applied Stirling's Approximation to the factorials on the right. Since $\lim_{g\to\infty}\frac{3g-4}{g} = 3$, we have

$$\left(\sum_{k=1}^{2g-3}\left(2^{k-1}\frac{(3g-4)!}{(3g-3-k)!}\right)\right) \sim \sqrt{3}(\frac{2}{e})^{2g-4}\frac{(3g-4)^{3g-4}}{g^g}.$$

We have $\lim_{g\to\infty}\frac{(3g-4)^{3g-4}}{(3g)^{3g-4}} = \frac{1}{e^4}$, so then

$$\begin{aligned}\left(\sum_{k=1}^{2g-3}\left(2^{k-1}\frac{(3g-4)!}{(3g-3-k)!}\right)\right) &\sim \sqrt{3}(\frac{2}{e})^{2g-4}\frac{1}{e^4}\frac{(3g)^{3g-4}}{g^g}\\ &= \sqrt{3}\frac{2^{2g-4}3^{3g-4}}{e^{2g}}g^{2g-4}\\ &= \frac{\sqrt{3}}{2^4 3^4}\left(\frac{2^2 3^3}{e^2}\right)^g g^{2g-4}.\end{aligned}$$

To simplify further, we have

$$\begin{aligned}\frac{\sqrt{3}}{2^4 3^4}\left(\frac{2^2 3^3}{e^2}\right)^g g^{2g-4} &\lesssim \left(\frac{2^2 3^3}{e^2}\right)^g g^{2g-4}\\ &\lesssim g^g g^{2g-4}\\ &= g^{3g-4}\\ &\lesssim g^{3g}.\end{aligned}$$

We now compute the asymptotics for the other component of the coefficient of $\mathrm{Sep}_1(g)$. Since $\lfloor f \rfloor \sim f$, we have

$$\begin{aligned}(2(3g-3))^{M_g-1} &= (2(3g-3))^{\lfloor 2\log_2(g-1)+1 \rfloor}\\ &\sim (2(3g-3))^{2\log_2(g-1)+1}.\end{aligned}$$

Since $\lim_{g\to\infty} \dfrac{\log_2(g-1)}{\log_2(g)} = 1$, we have

$$\begin{aligned}(2(3g-3))^{M_g-1} &\sim (2(3g-3))^{2\log_2(g)+1}\\ &= 2g^2(3g-3)^{2\log_2(g)+1}.\end{aligned}$$

We have $\lim_{g\to\infty} \dfrac{(3g-3)^{2\log_2(g)+1}}{(3g)^{2\log_2(g)+1}} = 1$, so then

$$\begin{aligned}(2(3g-3))^{M_g-1} &\sim 2g^2(3g)^{2\log_2(g)+1}\\ &= 6(9)^{\log_2(g)} g^{2\log_2(g)+3}.\end{aligned}$$

To simplify further, we have

$$\begin{aligned}6(9)^{\log_2(g)} g^{2\log_2(g)+3} &\lesssim (9)^{\log_2(g)} g^{2\log_2(g)+3}\\ &\lesssim g^{\log_2(g)} g^{2\log_2(g)+3}\\ &= g^{3\log_2(g)+3}.\end{aligned}$$

Then we have

$$\left(\sum_{k=1}^{2g-3}\left(2^{k-1}\frac{(3g-4)!}{(3g-3-k)!}\right)+(2(3g-3))^{M_g-1}\right)\lesssim g^{3g}+g^{3\log_2(g)+3}$$
$$\lesssim g^{3g}.$$

So

$$\operatorname{Tot}(g)\lesssim g^{3g}\operatorname{Sep}_1(g).$$

We do Theorem 17 next. Theorem 17 says

$$\operatorname{Tot}(g)\leq\left(\sum_{c=1}^{M_g}(2(3g-3))^{c-1}\right)(3(g-1)-3)\operatorname{Tot}(g-1).$$

We calculate asymptotics for the coefficient of $\operatorname{Tot}(g-1)$. The term in the sum with the fastest growth is the one that corresponds to $c=M_g$, so we have

$$\begin{aligned}\left(\sum_{c=1}^{M_g}(2(3g-3))^{c-1}\right)(3(g-1)-3)&\sim(2(3g-3))^{M_g-1}(3(g-1)-3)\\&=(2(3g-3))^{2\log_2(g)+1}(3(g-1)-3)\\&\sim 2^{2\log_2(g)+1}(3g-3)^{2\log_2(g)+2}\\&=24^{\log_2(g)}(3g-3)^{2\log_2(g)+2}\\&\sim 24^{\log_2(g)}(3g)^{2\log_2(g)+2}\\&=18(36)^{\log_2(g)}g^{2\log_2(g)+2}.\end{aligned}$$

Then we have

$$\begin{aligned}18(36)^{\log_2(g)}g^{2\log_2(g)+2}&\lesssim(36)^{\log_2(g)}g^{2\log_2(g)+2}\\&\lesssim(g)^{\log_2(g)}g^{2\log_2(g)+2}\\&=g^{3\log_2(g)+4}.\end{aligned}$$

So then we have

$$\frac{\operatorname{Tot}(g)}{\operatorname{Tot}(g-1)}\lesssim g^{3\log_2(g)+4}.$$

In Brinkmann, Goedgebeur and Van Cleemput [7], we have counts for the number of trivalent multigraphs for very small n, where $n = 2g - 2$ is the number of vertices. It appears that their data could grow like g^{cg} for some constant g.