Exploration of the Grothendieck-Teichmueller (GT) shadows for the dihedral poset

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Abstract

Grothendieck-Teichmueller (GT) shadows are morphisms of the groupoid GTSh and they may be thought of as approximations of elements of (the gentle version of) the Grothendieck-Teichmueller group $\widehat{\mathsf{GT}}$. The set $\mathrm{Ob}(\mathsf{GTSh})$ of objects of GTSh is the poset of certain finite index normal subgroups of the Artin braid group on 3 strands. In this note, we introduce a subposet Dih of $\mathrm{Ob}(\mathsf{GTSh})$, call it the dihedral poset, and investigate connected components of the groupoid GTSh for elements of this poset. We prove that every $\mathsf{K} \in \mathsf{Dih}$ is the only object of its connected component $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K})$ in the groupoid GTSh (in particular, $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K})$ is a finite group). We describe the set of morphisms of $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K})$ explicitly and we show that, for every pair $\mathsf{N}, \mathsf{K} \in \mathsf{Dih}$ such that $\mathsf{K} \le \mathsf{N}$, the natural map $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{K}) \to \mathsf{GTSh}_{\mathrm{conn}}(\mathsf{N})$ is surjective.

1 Introduction

In this paper, we explore a certain groupoid GTSh which is related to the gentle version[1] [7], [13] $\widehat{\mathsf{GT}}_{gen}$ of the Grothendieck-Teichmueller group $\widehat{\mathsf{GT}}$ [4, Section 4]. Many challenging questions [10], [11] about $\widehat{\mathsf{GT}}$, $\widehat{\mathsf{GT}}_{gen}$ and other versions of $\widehat{\mathsf{GT}}$ are motivated by a connection between $\widehat{\mathsf{GT}}$ and the absolute Galois group of rational numbers $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let $\widehat{\mathbb{Z}}$ (resp. $\widehat{\mathsf{F}}_2$) be the profinite completion of the ring $\mathbb{Z}$ (resp. the free group F_2 on two generators). The group $\widehat{\mathsf{GT}}_{gen}$ consists of pairs $(\hat{m}, \hat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\mathsf{F}}_2$ satisfying the hexagon relations (see equations (3.9), (3.10) in [13, Section 3.1]) and additional technical conditions. For the definition of the multiplication in $\widehat{\mathsf{GT}}_{gen}$, we refer the reader to [13, Section 3.1].

The group $\widehat{\mathsf{GT}}_{gen}$ receives a homomorphism Ih from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the form

$$\mathrm{Ih}(g) := \big((\chi(g) - 1)/2, f_g\big), \tag{1.1}$$

where $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^{\times}$ is the cyclotomic character and f_g is an element of $\widehat{\mathsf{F}}_2$ whose construction is described in [8, Section 1.4].

Belyi's theorem [1] implies[2] that the homomorphism Ih is injective and we call Ih the **Ihara embedding**.

[1] In [7], $\widehat{\mathsf{GT}}_{gen}$ is denoted by $\widehat{\mathsf{GT}}_0$

[2] See also Theorems 4.7.6, 4.7.7 and Fact 4.7.8 in [12].

1.1 The groupoid GTSh of GT-shadows in a nutshell

Let B_3 be the Artin braid group [9] on 3 strands:

$$\mathrm{B}_3 := \langle\, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle$$

and PB_3 be the kernel of the standard homomorphism ρ from B_3 to the symmetric group S_3 on 3 letters. It is known [9, Section 1.3] that

$$\mathrm{PB}_3 \cong \langle\, x_{12}, x_{23} \,\rangle \times \langle\, c \,\rangle,$$

where $x_{12} := \sigma_1^2$, $x_{23} := \sigma_2^2$ and $c := (\sigma_1\sigma_2\sigma_1)^2$. We identify the free group F_2 on two generators with the subgroup $\langle\, x_{12}, x_{23} \,\rangle$ of PB_3 generated by x_{12} and x_{23}.

Just as in [2], [13], we denote by GTSh the groupoid whose set $\mathrm{Ob}(\mathsf{GTSh})$ of objects is the poset

$$\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3) := \{\mathsf{N} \trianglelefteq \mathrm{B}_3 \mid \mathsf{N} \le \mathrm{PB}_3, \quad |\mathrm{PB}_3 : \mathsf{N}| < \infty\}.$$

For $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, we consider the finite set

$$\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \;\times\; \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}\,, \tag{1.2}$$

where $\mathsf{N}_{\mathsf{F}_2} := \mathsf{N} \cap \langle\, x_{12}, x_{23} \,\rangle$ and N_{ord} is the least common multiple of the orders of the cosets $x_{12}\mathsf{N}$, $x_{23}\mathsf{N}$ and $c\mathsf{N}$ in the finite group PB_3/N.

We denote by $\mathsf{GT}(\mathsf{N})$ the set of morphisms of the groupoid GTSh with the target N. These are elements of the finite set (1.2) that satisfy the hexagon relations (see (2.3), (2.4)) modulo N and additional technical conditions. We call morphisms of the groupoid GTSh **GT-shadows**.

Let (m, f) be a pair in $\mathbb{Z} \times \mathsf{F}_2$ that represents a GT-shadow with the target N. Hexagon relations (2.3), (2.4) imply that the formulas

$$T_{m,f}(\sigma_1) := \sigma_1^{2m+1}\,\mathsf{N}, \qquad T_{m,f}(\sigma_2) := f^{-1}\sigma_2^{2m+1} f\,\mathsf{N}$$

define a group homomorphism $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/\mathsf{N}$. It is convenient to denote by $[m, f]$ the element of $\mathsf{GT}(\mathsf{N})$ represented by a pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$.

For $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, the set $\mathsf{GTSh}(\mathsf{K}, \mathsf{N})$ of morphisms in GTSh from K to N is

$$\mathsf{GTSh}(\mathsf{K}, \mathsf{N}) := \{[m, f] \in \mathsf{GT}(\mathsf{N}) \mid \ker(T_{m,f}) = \mathsf{K}\}.$$

For the definition of the composition of morphisms in GTSh, we refer the reader to Theorem 2.14 of this paper (see also [13, Section 2.3]).

The groupoid GTSh is highly disconnected. Indeed, due to Proposition 2.12, if $\mathsf{GTSh}(\mathsf{K}, \mathsf{N})$ is non-empty, then the quotient groups B_3/N and B_3/K are isomorphic. However, using the finiteness of the set $\mathsf{GT}(\mathsf{N})$, it is easy to show that, for every $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, the connected component $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{N})$ of N in GTSh is a finite groupoid.

It is certainly easier to work with a connected component of GTSh that has exactly one object. Thus, if N is the only object of its connected component $\mathsf{GTSh}_{\mathrm{conn}}(\mathsf{N})$, then we say that N is an **isolated object** of GTSh. In this case, $\mathsf{GT}(\mathsf{N}) = \mathsf{GTSh}(\mathsf{N}, \mathsf{N})$ and hence $\mathsf{GT}(\mathsf{N})$ is a (finite) group.

Let H, K be elements of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ such that $\mathsf{H} \le \mathsf{K}$. Furthermore, let (m, f) be a pair in $\mathbb{Z} \times \mathsf{F}_2$ that represents a GT-shadow with the target H. Due to [13, Proposition 2.13], the same pair (m, f) also represents a GT-shadow with the target K and we get a natural map

$$\mathcal{R}_{\mathsf{H},\mathsf{K}} : \mathsf{GT}(\mathsf{H}) \to \mathsf{GT}(\mathsf{K}).$$

It is not hard to show that, if $\mathsf{H} \le \mathsf{K}$ are isolated objects of GTSh, then the map $\mathcal{R}_{\mathsf{H},\mathsf{K}}$ is a group homomorphism. In this paper, we call $\mathcal{R}_{\mathsf{H},\mathsf{K}}$ the **reduction map** and, sometimes, the **reduction homomorphism**.

1.2 A link between $\widehat{\mathsf{GT}}_{gen}$ and the groupoid GTSh

For a group G and a finite index normal subgroup N we denote by $\widehat{\mathcal{P}}_{\mathsf{N}}$ the standard group homomorphism

$$\widehat{\mathcal{P}}_{\mathsf{N}} : \widehat{G} \to G/\mathsf{N}$$

from the profinite completion $\widehat{G}$ of G to the finite group G/N. Moreover, for a positive integer N, we set $\widehat{\mathcal{P}}_N := \widehat{\mathcal{P}}_{N\mathbb{Z}}$, i.e. $\widehat{\mathcal{P}}_N$ is the standard ring homomorphism from $\widehat{\mathbb{Z}}$ to the finite ring $\mathbb{Z}/N\mathbb{Z}$.

Given $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and $(\hat{m}, \hat{f}) \in \widehat{\mathsf{GT}}_{gen}$, the pair

$$\big(\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{\mathsf{N}_{\mathsf{F}_2}}(\hat{f})\big)$$

is a GT-shadow with the target N. In other words, the formula

$$\mathscr{P}\mathscr{R}_{\mathsf{N}}(\hat{m}, \hat{f}) := \big(\widehat{\mathcal{P}}_{N_{\mathrm{ord}}}(\hat{m}), \widehat{\mathcal{P}}_{\mathsf{N}_{\mathsf{F}_2}}(\hat{f})\big)$$

defines a natural map $\mathscr{P}\mathscr{R}_{\mathsf{N}} : \widehat{\mathsf{GT}}_{gen} \to \mathsf{GT}(\mathsf{N})$. If a GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ belongs to the image of $\mathscr{P}\mathscr{R}_{\mathsf{N}}$, then we say that $[m, f]$ is **genuine**; otherwise $[m, f]$ is called **fake**.

One can show [2] that a GT-shadow $[m, f] \in \mathsf{GT}(\mathsf{N})$ is genuine if and only if $[m, f]$ belongs to the image of the reduction map $\mathcal{R}_{\mathsf{H},\mathsf{N}}$ for every $\mathsf{H} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ such that $\mathsf{H} \le \mathsf{N}$. At the time of writing, the authors (as well as the mentor), do not know a single example of a fake GT-shadow.

Remark 1.1 Using the reduction maps, one can construct [2] a functor $\mathcal{ML}$ from the subposet

$$\mathsf{NFI}^{isolated}_{\mathrm{PB}_3}(\mathrm{B}_3) \subset \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$$

of isolated objects of the groupoid GTSh to the category of finite groups. Moreover, one can show [2] that the natural group homomorphism $\widehat{\mathsf{GT}}_{gen} \to \lim(\mathcal{ML})$ is an isomorphism of (topological) groups.

Remark 1.2 In papers [5] and [6] by P. Guillot, the author investigated a similar construction related to the group $\widehat{\mathsf{GT}}_{gen}$. He used an equivalent by quite different definition of $\widehat{\mathsf{GT}}_{gen}$ (see [7, Main Theorem, (a)]).

Remark 1.3 GT-shadows for the original version of $\widehat{\mathsf{GT}}$ [4, Section 4] were introduced in paper [3]. Note that, in paper [3], the notation GTSh is used for the groupoid of GT-shadows for $\widehat{\mathsf{GT}}$ and the set of objects of this groupoid is $\mathsf{NFI}_{\mathrm{PB}_4}(\mathrm{B}_4)$. In this paper, GTSh denotes the groupoid of GT-shadows for $\widehat{\mathsf{GT}}_{gen}$ and, here, $\mathrm{Ob}(\mathsf{GTSh}) := \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

1.3 The dihedral poset and the results of the paper

In this paper, we introduce a natural subposet $\mathsf{Dih} \subset \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. More precisely, for every $n \in \mathbb{Z}_{\geq 3}$, we denote by ψ_n the following group homomorphism $\psi_n : \mathrm{PB}_3 \to D_n \times D_n \times D_n$

$$\psi_n(x_{12}) := (r, s, s), \qquad \psi_n(x_{23}) := (rs, r, rs), \qquad \psi_n(c) := (1, 1, 1),$$

where D_n is the dihedral group $\langle\, r, s \mid r^n,\ s^2,\ rsrs \,\rangle$ of order $2n$.

Due to Proposition 3.1, the subgroup $\mathsf{K}^{(n)} := \ker(\psi_n)$ is an element of the poset $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. So we set

$$\mathsf{Dih} := \{\mathsf{K}^{(n)} \mid n \in \mathbb{Z}_{\geq 3}\}$$

and call Dih the **dihedral poset**.

The main result of this paper is Theorem 4.3. The first statement of this theorem gives us an explicit description of the set $\mathsf{GT}(\mathsf{K})$ for every $\mathsf{K} \in \mathsf{Dih}$. Due to the second statement of this theorem, every $\mathsf{K} \in \mathsf{Dih}$ is the only object of its connected component in the groupoid GTSh. In particular, $\mathsf{GT}(\mathsf{K})$ is a (finite) group for every $\mathsf{K} \in \mathsf{Dih}$.

In this paper, we also prove that, for every pair $\mathsf{H}, \mathsf{K} \in \mathsf{Dih}$ such that $\mathsf{H} \leq \mathsf{K}$, the reduction map

$$\mathcal{R}_{\mathsf{H},\mathsf{K}} : \mathsf{GT}(\mathsf{H}) \to \mathsf{GT}(\mathsf{K})$$

is surjective (see Theorem 4.7). This implies that one cannot find an example of a fake GT-shadow using only the dihedral poset Dih.

Organization of the paper. In Section 2, we give a brief reminder of the groupoid GTSh of GT-shadows. In this section, we recall many statements from [13]. In Section 3, we introduce a subposet $\mathsf{Dih} \subset \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, called the dihedral poset. In this section, we also give an explicit description of the commutator subgroup $[\mathsf{F}_2/\mathsf{K}_{\mathsf{F}_2}, \mathsf{F}_2/\mathsf{K}_{\mathsf{F}_2}]$ for $\mathsf{K} \in \mathsf{Dih}$. In Section 4, we describe the set $\mathsf{GT}(\mathsf{K})$ of GT-shadows for an arbitrary element $\mathsf{K} \in \mathsf{Dih}$. This description is presented in Theorem 4.3. Due to the same theorem, every $\mathsf{K} \in \mathsf{Dih}$ is an isolated object of the groupoid GTSh. At the end of Section 4, we prove that the reduction map $\mathcal{R}_{\mathsf{H},\mathsf{K}} : \mathsf{GT}(\mathsf{H}) \to \mathsf{GT}(\mathsf{K})$ is surjective for every pair $\mathsf{H}, \mathsf{K} \in \mathsf{Dih}$ with $\mathsf{H} \leq \mathsf{K}$ (see Theorem 4.7).

1.4 Notational conventions

For a set X with an equivalence relation and $a \in X$ we will denote by $[a]$ the equivalence class which contains the element a. The notation gcd (resp. lcm) is reserved for the greatest common divisor (resp. the least common multiple). C_n denotes the cyclic group of order n.

The notation B_n (resp. PB_n) is reserved for the Artin braid group on n strands (resp. the pure braid group on n strands). S_n denotes the symmetric group on n letters. We denote by σ_1 and σ_2 the standard generators of B_3. Furthermore, we denote by x_{12}, x_{23} and x_{13} the standard generators of PB_3

$$x_{12} := \sigma_1^2, \qquad x_{23} := \sigma_2^2, \qquad x_{13} := \sigma_2 \sigma_1^2 \sigma_2^{-1}\,. \tag{1.3}$$

We recall that

$$c := x_{23} x_{12} x_{13} = x_{12} x_{13} x_{23} = (\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3 \tag{1.4}$$

belongs to the center $\mathcal{Z}(\mathrm{PB}_3)$ of PB_3 (and the center $\mathcal{Z}(\mathrm{B}_3)$ of B_3). Moreover, $\mathcal{Z}(\mathrm{B}_3) = \mathcal{Z}(\mathrm{PB}_3) = \langle\, c \,\rangle \cong \mathbb{Z}$.

We denote by Δ the following element of B_3

$$\Delta := \sigma_1\sigma_2\sigma_1 \tag{1.5}$$

and observe that

$$\sigma_1\Delta = \Delta\sigma_2, \qquad \sigma_2\Delta = \Delta\sigma_1, \qquad \sigma_1^{-1}\Delta = \Delta\sigma_2^{-1}, \qquad \sigma_2^{-1}\Delta = \Delta\sigma_1^{-1}, \tag{1.6}$$

$$\Delta^2 = c. \tag{1.7}$$

Using identities (1.6) and (1.7), it is easy to see that the adjoint action of B_3 on PB_3 is given on generators by the formulas:

$$\sigma_1 x_{12}\sigma_1^{-1} = \sigma_1^{-1}x_{12}\sigma_1 = x_{12}, \qquad \sigma_1 x_{23}\sigma_1^{-1} = x_{23}^{-1}x_{12}^{-1}c, \qquad \sigma_1^{-1}x_{23}\sigma_1 = x_{12}^{-1}x_{23}^{-1}c, \tag{1.8}$$

$$\sigma_2 x_{12}\sigma_2^{-1} = x_{12}^{-1}x_{23}^{-1}c, \qquad \sigma_2^{-1}x_{12}\sigma_2 = x_{23}^{-1}x_{12}^{-1}c \qquad \sigma_2 x_{23}\sigma_2^{-1} = \sigma_2^{-1}x_{23}\sigma_2 = x_{23}\,. \tag{1.9}$$

Moreover,

$$\Delta x_{12}\Delta^{-1} = x_{23}, \qquad \Delta x_{23}\Delta^{-1} = x_{12}\,. \tag{1.10}$$

It is known that $\langle\, x_{12}, x_{23}\,\rangle$ is isomorphic to the free group F_2 on two generators and we tacitly identify F_2 with the subgroup $\langle\, x_{12}, x_{23}\,\rangle$ of PB_3. It is known [9, Section 1.3] that $PB_3 \cong \mathsf{F}_2 \times \mathbb{Z}$. We often use the following notation for x_{12}, x_{23} and $(x_{12}x_{23})^{-1}$:

$$x := x_{12}, \qquad y := x_{23}, \qquad z := y^{-1}x^{-1}\,.$$

We denote by θ and τ the automorphisms of $\mathsf{F}_2 := \langle\, x, y\,\rangle$ defined by the formulas

$$\theta(x) := y, \qquad \theta(y) := x, \tag{1.11}$$

$$\tau(x) := y, \qquad \tau(y) := y^{-1}x^{-1}\,. \tag{1.12}$$

For a group G, $\mathsf{End}(G)$ is the monoid of endomorphisms $G \to G$ and the notation $[G,G]$ is reserved for the commutator subgroup of G. For a subgroup $H \le G$, the notation $|G:H|$ is reserved for the index of H in G. For a normal subgroup $H \trianglelefteq G$ of finite index, we denote by $\mathsf{NFI}_H(G)$ the poset of finite index normal subgroups N in G such that $\mathsf{N} \le H$. Moreover, $\mathsf{NFI}(G) := \mathsf{NFI}_G(G)$, i.e. $\mathsf{NFI}(G)$ is the poset of normal finite index subgroups of a group G. For a subgroup $H \le G$, $\mathrm{Core}_G(H)$ denotes the normal core of H in G, i.e.

$$\mathrm{Core}_G(H) := \bigcap_{g\in G} gHg^{-1}\,.$$

For a finite group G, $|G|$ denotes the order of G.

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2 Reminder of the groupoid GTSh

Definition 2.1 *(N_{ord} and $\mathsf{N}_{\mathsf{F}_2}$) Let $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and let us define*

$$N_{\mathrm{ord}} := \mathrm{lcm}(\mathrm{ord}(x_{12}\mathsf{N}), \mathrm{ord}(x_{23}\mathsf{N}), ord(c\mathsf{N})). \tag{2.1}$$

Let also $\mathsf{F}_2 = \langle x, y \rangle$, where $x = x_{12}$, $y = x_{23}$ and

$$\mathsf{N}_{\mathsf{F}_2} := \mathsf{N} \cap \mathsf{F}_2. \tag{2.2}$$

Remark 2.2 Clearly, $\mathsf{N}_{\mathsf{F}_2} \in \mathsf{NFI}(\mathsf{F}_2)$.

Definition 2.3 *A GT-**pair with the target** N is a pair*

$$(m + N_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}_{\mathsf{F}_2}) \in \mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$$

satisfying the relations

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f\, \mathsf{N} = f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m\, \mathsf{N} \tag{2.3}$$

and

$$f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1}\, \mathsf{N} = \sigma_2 \sigma_1 x_{23}^{-m} c^m f\, \mathsf{N}. \tag{2.4}$$

*These relations are called the **hexagon relations**.*

It is easy to see from definitions of N_{ord} and $\mathsf{N}_{\mathsf{F}_2}$ that if a pair (m, f) satisfies the hexagon relations then all elements of the coset $(m + N_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}_{\mathsf{F}_2})$ satisfy the hexagon relations.

Definition 2.4 *A GT-pair with the target N is called **charming** if*

$$\gcd(2m+1, N_{\mathrm{ord}}) = 1 \quad \text{and} \quad f\mathsf{N}_{\mathsf{F}_2} \in [\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}, \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}].$$

Remark 2.5 We denote by

1. $\mathsf{GT}_{pr}(\mathsf{N})$ the set of GT-pairs with the target N;
2. $\mathsf{GT}_{pr}^{\heartsuit}(\mathsf{N})$ the set of <u>charming</u> GT-pairs with the target N;
3. $[m, f]$ the element of $\mathbb{Z}/N_{\mathrm{ord}}\mathbb{Z} \times \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$ represented by a pair $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$.

Proposition 2.6 *For every $[m, f] \in \mathsf{GT}_{pr}(\mathsf{N})$, the formulas*

$$T_{m,f}(\sigma_1) = \sigma_1^{2m+1}\, \mathsf{N}, \qquad T_{m,f}(\sigma_2) = f^{-1} \sigma_2^{2m+1} f\, \mathsf{N} \tag{2.5}$$

define a group homomorphism from B_3 to B_3/N.

Proof. It suffices to check that

$$T_{m,f}(\sigma_1) T_{m,f}(\sigma_2) T_{m,f}(\sigma_1) = T_{m,f}(\sigma_2) T_{m,f}(\sigma_1) T_{m,f}(\sigma_2).$$

Using normality of N, we can rewrite it as

$$\sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} \mathsf{N} = f^{-1} \sigma_2^{2m+1} f \sigma_1^{2m+1} f^{-1} \sigma_2^{2m+1} f \mathsf{N}. \tag{2.6}$$

Applying the first hexagon relation (2.3) to the left hand side of (2.6), we get

$$(\sigma_1^{2m+1}f^{-1}\sigma_2^{2m+1}f)\sigma_1^{2m+1}\mathsf{N} = f^{-1}\sigma_1\sigma_2 x_{12}^{-m}c^m\sigma_1^{2m+1}\mathsf{N}.$$

Recall that c commutes with all elements of B_3, $\Delta := \sigma_1\sigma_2\sigma_1$, and $x_{12} := \sigma_1^2$. Then the left hand side of (2.6) can be simplified further as

$$f^{-1}\sigma_1\sigma_2 x_{12}^{-m}c^m\sigma_1^{2m+1}\,\mathsf{N} = f^{-1}\sigma_1\sigma_2\sigma_1^{-2m}\sigma_1^{2m+1}c^m\,\mathsf{N} = f^{-1}\Delta c^m\,\mathsf{N}.$$

Now consider the right hand side of (2.6) and apply the first hexagon relation (2.3) to it. Using $\sigma_2\Delta = \Delta\sigma_1$, we obtain

$$f^{-1}\sigma_2^{2m+1}f(\sigma_1^{2m+1}f^{-1}\sigma_2^{2m+1}f)\,\mathsf{N} = f^{-1}\sigma_2^{2m+1}ff^{-1}\sigma_1\sigma_2x_{12}^{-m}c^m\,\mathsf{N} = f^{-1}\sigma_2^{2m}\sigma_2\sigma_1\sigma_2\sigma_1^{-2m}c^m\,\mathsf{N} =$$

$$= f^{-1}\sigma_2^{2m}\Delta\sigma_1^{-2m}c^m\mathsf{N} = f^{-1}\Delta\sigma_1^{2m}\sigma_1^{-2m}c^m\mathsf{N} = f^{-1}\Delta c^m\mathsf{N}.$$

Thus, equation (2.6) holds and $T_{m,f}$ is indeed a group homomorphism from B_3 to B_3/N. □

Recall [13, Proposition 2.6]:

Proposition 2.7 *Let* $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ *and* θ *and* τ *be the automorphisms of* F_2 *defined in* (1.11) *and* (1.12)*, respectively. A pair* $(m, f) \in \mathbb{Z}\times[\mathsf{F}_2, \mathsf{F}_2]$ *satisfies hexagon relations modulo* N *if and only if*

$$f\theta(f) \in \mathsf{N}_{\mathsf{F}_2} \tag{2.7}$$

and

$$\tau^2(y^m f)\tau(y^m f)y^m f \in \mathsf{N}_{\mathsf{F}_2}. \tag{2.8}$$

*We will call these two relations the **simplified hexagon relations**.*

Proposition 2.8 *We can restrict* $T_{m,f}$ *to* PB_3 *and define in such way a group homomorphism* $T_{m,f}^{\mathrm{PB}_3} : \mathrm{PB}_3 \to \mathrm{PB}_3/\mathsf{N}$.

Proof. It is enough to prove that $T_{m,f}(\mathrm{PB}_3) \subset \mathrm{PB}_3/\mathsf{N}$. Let us denote by ρ the standard homomorphism $\mathrm{B}_3 \to S_3 : \rho(\sigma_1) := (1, 2),\ \rho(\sigma_2) := (2, 3)$. As $\mathsf{N} \leqslant \mathrm{PB}_3$, the formula $\rho_{\mathsf{N}}(wN) := \rho(w)$ defines the group homomorphism

$$\rho_{\mathsf{N}} : \mathrm{B}_3/\mathsf{N} \to S_3.$$

It is easy to see that, for every $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and $[m, f] \in \mathsf{GT}_{pr}(\mathsf{N})$,

$$\rho_N \circ T_{m,f} = \rho$$

and hence $T_{m,f}(\mathrm{PB}_3) \subset \mathrm{PB}_3/\mathsf{N}$. □

Notice that

$$\ker(T_{m,f}) = \ker(T_{m,f}^{\mathrm{PB}_3}) \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3).$$

Similarly, for every $[m, f] \in \mathsf{GT}_{pr}(\mathsf{N})$, we can restrict $T_{m,f}^{\mathrm{PB}_3}$ to F_2 and obtain a group homomorphism $T_{m,f}^{\mathsf{F}_2} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$.

Recall [13, Proposition 2.7]:

Proposition 2.9 *If a pair* $(m, f) \in \mathbb{Z}\times\mathsf{F}_2$ *satisfies hexagon relations and* $\gcd(2m+1, N_{\mathrm{ord}}) = 1$*, then the following conditions are equivalent:*

1. *The homomorphism* $T_{m,f}$ *is surjective;*

2. *The homomorphism* $T^{\mathrm{PB}_3}_{m,f}$ *is surjective;*

3. *The homomorphism* $T^{\mathsf{F}_2}_{m,f}$ *is surjective.*

Definition 2.10 *A charming* GT*-pair* $[m,f]$ *is called a* GT*-**shadow with the target*** N *if the pair* (m,f) *satisfies one of the three conditions of the previous proposition.*

Remark 2.11 We denote by $\mathsf{GT}(\mathsf{N})$ the set of GT-shadows with the target N.

Since, for every $[m,f] \in \mathsf{GT}(\mathsf{N})$, the group homomorphisms $T_{m,f} : \mathrm{B}_3 \to \mathrm{B}_3/\mathsf{N}$ $T^{\mathrm{PB}_3}_{m,f} : \mathrm{PB}_3 \to \mathrm{PB}_3/\mathsf{N}$ and $T^{\mathsf{F}_2}_{m,f} : \mathsf{F}_2 \to \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$ are onto, $T_{m,f}$, $T^{\mathrm{PB}_3}_{m,f}$, and $T^{\mathsf{F}_2}_{m,f}$ induce the isomorphisms

$$T^{\mathrm{isom}}_{m,f} : \mathrm{B}_3/\mathsf{K} \xrightarrow{\simeq} \mathrm{B}_3/\mathsf{N}, \quad T^{\mathrm{PB}_3,\mathrm{isom}}_{m,f} : \mathrm{B}_3/\mathsf{K} \xrightarrow{\simeq} \mathrm{B}_3/\mathsf{N}, \quad T^{\mathsf{F}_2,\mathrm{isom}}_{m,f} : \mathsf{F}_2/\mathsf{K}_{\mathsf{F}_2} \xrightarrow{\simeq} \mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2},$$

respectively, where $\mathsf{K} := \ker T_{m,f}$.

This observation implies the first three statements of the following proposition[3]:

Proposition 2.12 *Let* $\mathsf{K}, \mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$*. If there exists* $[m,f] \in \mathsf{GT}(\mathsf{N})$ *such that* $\mathsf{K} = \ker(T_{m,f})$ *then*

1. *the finite groups* B_3/N *and* B_3/K *are isomorphic (and hence* $|\mathrm{B}_3 : \mathsf{N}| = |\mathrm{B}_3 : \mathsf{K}|$*);*

2. *the finite groups* PB_3/N *and* PB_3/K *are isomorphic (and hence* $|\mathrm{PB}_3 : \mathsf{N}| = |\mathrm{PB}_3 : \mathsf{K}|$*);*

3. *the finite groups* $\mathsf{F}_2/\mathsf{N}_{\mathsf{F}_2}$ *and* $\mathsf{F}_2/\mathsf{K}_{\mathsf{F}_2}$ *are isomorphic (and hence* $|\mathsf{F}_2 : \mathsf{N}_{\mathsf{F}_2}| = |\mathsf{F}_2 : \mathsf{K}_{\mathsf{F}_2}|$*);*

4. $\mathsf{K}_{\mathrm{ord}} = \mathsf{N}_{\mathrm{ord}}$.

We will now recall that GT-shadows form a groupoid GTSh with $\mathrm{Ob}(\mathsf{GTSh}) := \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$ and

$$\mathsf{GTSh}(\mathsf{K},\mathsf{N}) := \{[m,f] \in \mathsf{GT}(\mathsf{N}) \mid \ker(T_{m,f}) = \mathsf{K}\}.$$

Just as in [13, Section 2.3], for $(m,f) \in \mathbb{Z} \times \mathsf{F}_2$, we denote by $E_{m,f}$ the following endomorphism of F_2:

$$E_{m,f}(x) = x^{2m+1}, \quad E_{m,f}(y) = f^{-1} y^{2m+1} f.$$

Recall [13, Section 2.3] that

1. $E_{m_1,f_1} \circ E_{m_2,f_2} = E_{m,f}$, where $m := 2m_1m_2 + m_1 + m_2$ and $f = f_1 E_{m_1,f_1}(f_2)$;

2. $\mathbb{Z} \times \mathsf{F}_2$ is a monoid with the respect to the binary operation

$$(m_1, f_1) \bullet (m_2, f_2) = (2m_1m_2 + m_1 + m_2, f_1 E_{m_1,f_1}(f_2))$$

   and the identity element $(0, 1_{\mathsf{F}_2})$;

3. The assignment $(m,f) \to E_{m,f}$ defines a homomorphism of monoids

$$(\mathbb{Z} \times \mathsf{F}_2, \bullet) \to \mathsf{End}(\mathsf{F}_2).$$

[3] See also [13, Proposition 2.10].

Furthermore, if $(m, f) \in \mathbb{Z} \times \mathsf{F}_2$ represents a GT-pair with the target $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, then

$$T^{\mathsf{F}_2}_{m,f}(w) = E_{m,f}(w)\mathsf{N}_{\mathsf{F}_2}, \quad \forall\, w \in \mathsf{F}_2.$$

Due to the following two statements that "unpack" [13, Theorem 2.12], GTSh is indeed a groupoid.

Proposition 2.13 *Let* $\mathsf{N}^{(1)}, \mathsf{N}^{(2)}, \mathsf{N}^{(3)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3), [m_1, f_1] \in \mathsf{GTSh}(\mathsf{N}^{(2)}, \mathsf{N}^{(1)}), [m_2, f_2] \in \mathsf{GTSh}(\mathsf{N}^{(3)}, \mathsf{N}^{(2)})$ *and* $\mathsf{N}_{\mathrm{ord}} := \mathsf{N}^{(1)}_{\mathrm{ord}} = \mathsf{N}^{(2)}_{\mathrm{ord}} = \mathsf{N}^{(3)}_{\mathrm{ord}}$. *If*

$$m := 2m_1m_2 + m_1 + m_2 \quad \text{and} \quad f = f_1 E_{m_1,f_1}(f_2)$$

then

$$(m + \mathsf{N}_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}^{(1)}_{\mathsf{F}_2}) \in \mathsf{GTSh}(\mathsf{N}^{(3)}, \mathsf{N}^{(1)}).$$

The pair $[m, f] := (m + \mathsf{N}_{\mathrm{ord}}\mathbb{Z}, f\mathsf{N}^{(1)}_{\mathsf{F}_2})$ *depends only on the cosets* $f_1\mathsf{N}^{(1)}$, $f_2\mathsf{N}^{(2)}$ *and residue classes* $m_1 + \mathsf{N}_{\mathrm{ord}}\mathbb{Z}$, $m_2 + \mathsf{N}_{\mathrm{ord}}\mathbb{Z}$. *Moreover,*

$$T^{\mathrm{isom}}_{m_1,f_1} \circ T^{\mathrm{isom}}_{m_2,f_2} = T^{\mathrm{isom}}_{m,f}.$$

Theorem 2.14 *Let* $\mathsf{N}^{(1)}, \mathsf{N}^{(2)}, \mathsf{N}^{(3)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, $[m_1, f_1] \in \mathsf{GTSh}(\mathsf{N}^{(2)}, \mathsf{N}^{(1)})$, $[m_2, f_2] \in \mathsf{GTSh}(\mathsf{N}^{(3)}, \mathsf{N}^{(2)})$ *and* $\mathsf{N}_{\mathrm{ord}} := \mathsf{N}^{(1)}_{\mathrm{ord}} = \mathsf{N}^{(2)}_{\mathrm{ord}} = \mathsf{N}^{(3)}_{\mathrm{ord}}$.

1. *Then the formula*

$$[m_1, f_1] \circ [m_2, f_2] = [2m_1m_2 + m_1 + m_2, f_1 E_{m_1,f_1}(f_2)] \tag{2.9}$$

   defines a composition of morphisms in GTSh;

2. *For every* $\mathsf{N} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$, *the pair* $(0, 1_{\mathsf{F}_2})$ *represents the identity morphism in* $\mathsf{GTSh}(\mathsf{N}, \mathsf{N})$;

3. *Finally, for every* $[m, f] \in \mathsf{GTSh}(\mathsf{K}, \mathsf{N})$, *the formulas*

$$\tilde{m} + \mathsf{N}_{\mathrm{ord}}\mathbb{Z} := -(2\overline{m} + 1)^{-1}\overline{m}, \quad \tilde{f}\mathsf{K}_{\mathsf{F}_2} := (T^{\mathsf{F}_2,\mathrm{isom}}_{m,f})^{-1}(f^{-1}\mathsf{N}_{\mathsf{F}_2}) \tag{2.10}$$

   define the inverse $[\tilde{m}, \tilde{f}] \in \mathsf{GTSh}(\mathsf{N}, \mathsf{K})$ *of the morphism* $[m, f]$.

3 The dihedral poset

Let $n \in \mathbb{Z}_{\geq 3}$ and $D_n := \langle\, r, s \mid r^n,\ s^2,\ srs^{-1}r \,\rangle$. The starting point of the story is the group homomorphism $\psi_n : \mathrm{PB}_3 \to D_n^3$ defined by the formulas:

$$\psi_n(x_{12}) := (r, s, s), \qquad \psi_n(x_{23}) := (rs, r, rs), \qquad \psi_n(c) := (1, 1, 1). \tag{3.1}$$

We set $\mathsf{K}^{(n)} := \ker(\psi_n)$ and we claim that

Proposition 3.1 *For every* $n \in \mathbb{Z}_{\geq 3}$, $\mathsf{K}^{(n)}$ *belongs to the poset* $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$.

Proof. First, note that $\mathsf{K}^{(n)}$ is a finite index subgroup of PB_3 because D_n^3 is finite. The subgroup PB_3 also has finite index in B_3, so $\mathsf{K}^{(n)}$ has finite index in B_3. Thus it remains to show that $\mathsf{K}^{(n)}$ is normal in B_3.

Consider the map $\varphi : \mathrm{PB}_3 \to D_n$ given by

$$\varphi(x_{12}) := s, \qquad \varphi(x_{23}) := rs, \qquad \varphi(c) := 1.$$

We will show that $\mathsf{K}^{(n)}$ is the normal core in B_3 of $\ker\varphi \leq \mathrm{PB}_3$. Define for $w \in \mathrm{B}_3$ the map $\varphi^w : \mathrm{PB}_3 \to D_n$ given by

$$\varphi^w(g) := \varphi(w^{-1}gw), \qquad g \in \mathrm{PB}_3.$$

Note that

$$\ker(\varphi^w) = w\ker(\varphi)w^{-1},$$

and hence

$$C := \mathrm{Core}_{\mathrm{B}_3}(\ker\varphi) = \bigcap_{w\in\mathrm{B}_3} \ker(\varphi^w).$$

Since $|\mathrm{B}_3 : \mathrm{PB}_3| = 6$ and that the elements

$$1,\ \ \sigma_1^{-1},\ \ \sigma_2^{-1},\ \ \Delta^{-1},\ \ \sigma_1^{-1}\Delta^{-1},\ \ \sigma_2^{-1}\Delta^{-1}$$

form a complete set of coset representatives, we have

$$C = \ker(\varphi) \cap \ker(\varphi^{\sigma_1^{-1}}) \cap \ker(\varphi^{\sigma_2^{-1}}) \cap \ker(\varphi^{\Delta^{-1}}) \cap \ker(\varphi^{\sigma_1^{-1}\Delta^{-1}}) \cap \ker(\varphi^{\sigma_2^{-1}\Delta^{-1}}). \tag{3.2}$$

We will now show that

$$C = \ker\varphi \cap \ker(\varphi^{\sigma_1^{-1}}) \cap \ker(\varphi^{\sigma_2^{-1}}). \tag{3.3}$$

Let γ be the following automorphism of D_n

$$\gamma(r) := r^{-1}, \qquad \gamma(s) := rs.$$

Clearly, $\gamma(s) = rs$ and $\gamma(rs) = s$ or equivalently $\gamma\circ\varphi(x_{12}) = \varphi(x_{23})$ and $\gamma\circ\varphi(x_{23}) = \varphi(x_{12})$. Since conjugation by Δ swaps x_{12} and x_{23} and $\varphi(c) = 1$, we have

$$\varphi^{w\Delta^{-1}}(g) = \gamma\circ\varphi^w(g), \quad \forall\ g \in \mathrm{PB}_3,\ w \in \mathrm{B}_3\,.$$

This gives us

$$\ker\varphi = \ker(\varphi^{\Delta^{-1}}), \qquad \ker(\varphi^{\sigma_1^{-1}}) = \ker(\varphi^{\sigma_1^{-1}\Delta^{-1}}), \qquad \ker(\varphi^{\sigma_2^{-1}}) = \ker(\varphi^{\sigma_2^{-1}\Delta^{-1}}),$$

which proves (3.3).

Let $\tilde\psi := \varphi^{\sigma_2^{-1}} \times \varphi^{\sigma_1^{-1}} \times \varphi : \mathrm{PB}_3 \to D_n^3$. Using (1.8) and (1.9) we see that

$$\tilde\psi(x_{12}) = (r^{-1}, s, s), \qquad \tilde\psi(x_{23}) = (rs, r, rs), \qquad \tilde\psi(c) = (1,1,1).$$

Identity (3.3) implies that $C = \ker(\tilde\psi)$.

Let j be the following inner automorphism of D_n^3:

$$j(g_1, g_2, g_3) := (rs(g_1)(rs)^{-1}, g_2, g_3).$$

Since $\psi_n = j \circ \tilde{\psi}$, we have

$$\mathsf{K}^{(n)} := \ker(\psi_n) = \ker(\tilde{\psi}) = C.$$

Since $\mathsf{K}^{(n)} \trianglelefteq \mathrm{B}_3$, this completes the proof that $\mathsf{K}^{(n)} \in \mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$. □

We denote by Dih the subposet of $\mathsf{NFI}_{\mathrm{PB}_3}(\mathrm{B}_3)$

$$\mathsf{Dih} := \{\mathsf{K}^{(n)} \mid n \in \mathbb{Z}_{\geq 3}\}$$

and call it the **dihedral poset**.

Remark 3.2 For every $n \in \mathbb{Z}_{\geq 3}$, $\mathsf{K}^{(n)}_{\mathsf{F}_2}$ is the kernel of the homomorphism $\mathsf{F}_2 \to D_n^3$ that sends x to (r, s, s) and y to (rs, r, rs). Moreover,

$$K^{(n)}_{\mathrm{ord}} = \mathrm{lcm}(n, 2). \tag{3.4}$$

Remark 3.3 If $q, n \in \mathbb{Z}_{\geq 3}$, $n \mid q$, and $D_q = \langle\, a, b \mid a^q,\ b^2,\ bab^{-1}a\,\rangle$, then the formulas

$$\eta_{q,n}(a) := r, \qquad \eta_{q,n}(b) := s \tag{3.5}$$

define a natural homomorphism $\eta_{q,n} : D_q \to D_n$. Since $\eta^3_{q,n} \circ \psi_q = \psi_n$, we have $\mathsf{K}^{(q)} \leq \mathsf{K}^{(n)}$.

It is convenient to identify $\mathsf{F}_2/\mathsf{K}^{(n)}_{\mathsf{F}_2}$ with the subgroup

$$G_n := \langle\, (r, s, s), (rs, r, rs)\,\rangle \leq D_n^3\,.$$

For $w \in \mathsf{F}_2$, $\overline{w}$ denotes the coset $w\mathsf{K}^{(n)}_{\mathsf{F}_2}$. Thus,

$$\overline{x} = (r, s, s), \qquad \overline{y} = (rs, r, rs), \qquad \overline{z} = (r^2 s, r^{-1} s, r). \tag{3.6}$$

Due to this identification and Proposition 2.7, the set $\mathsf{GT}^{\heartsuit}_{pr}(\mathsf{K}^{(n)})$ of charming GT-pairs is identified with the set of pairs

$$(m, g) \ \in\ \{0, 1, \dots, K^{(n)}_{\mathrm{ord}} - 1\} \times [G_n, G_n]$$

for which $\gcd(2m+1, K^{(n)}_{\mathrm{ord}}) = 1$,

$$g\theta(g) = 1 \tag{3.7}$$

and

$$\tau^2(\overline{y}^m g)\tau(\overline{y}^m g)\overline{y}^m g = 1. \tag{3.8}$$

3.1 The description of the commutator subgroup $[G_n, G_n]$

To proceed with description of the set of the GT-shadows with the target $\mathsf{K}^{(n)}$, it is useful have some information about the commutator subgroup of G_n. So let us prove the following proposition:

Proposition 3.4 *For every $n \in \mathbb{Z}_{\geq 3}$, the commutator subgroup $[G_n, G_n]$ of $G_n := \langle\, \overline{x}, \overline{y}\,\rangle$ consists of elements of the form*

$$(r^{2n_1}, r^{2n_2}, r^{2n_3}), \qquad (n_1, n_2, n_3) \in (2\mathbb{Z})^3 \ \ \text{or}\ \ (n_1, n_2, n_3) \in (2\mathbb{Z}+1)^3 \tag{3.9}$$

i.e. n_1, n_2, n_3 are either all even integers or all odd integers.

Proof. It is easy to see that the subset

$$C_n := \{(r^{2n_1}, r^{2n_2}, r^{2n_3}) \mid (n_1, n_2, n_3) \in (2\mathbb{Z})^3 \;\text{ or }\; (n_1, n_2, n_3) \in (2\mathbb{Z}+1)^3\} \;\subset\; G_n$$

is a (normal) subgroup of G_n.

Since G_n is generated by two elements and the commutator subgroup $[\mathsf{F}_2, \mathsf{F}_2]$ of F_2 is generated by elements of the form

$$[x^t, y^h] = x^t y^h x^{-t} y^{-h}\,, \qquad t, h \in \mathbb{Z}, \tag{3.10}$$

we conclude that $[G_n, G_n]$ is generated by the elements

$$[\overline{x}^t, \overline{y}^h], \qquad t, h \in \mathbb{Z}. \tag{3.11}$$

We need to consider 4 cases: t, h are even, t is even and h is odd, t is odd and h is even, t, h are odd.

If t, h are even, then $[\overline{x}^t, \overline{y}^h] = (1, 1, 1)$.

If t is odd and h is even, then we get

$$\overline{x}^t \overline{y}^h \overline{x}^{-t} \overline{y}^{-h} = (r^t, s, s)(1, r^h, 1)(r^{-t}, s, s)(1, r^{-h}, 1) = (1, [s, r^h], 1) = (1, r^{2h}, 1).$$

If t is even and h is odd, then we get

$$\overline{x}^t \overline{y}^h \overline{x}^{-t} \overline{y}^{-h} = (r^t, 1, 1)(rs, r^h, rs)(r^{-t}, 1, 1)(rs, r^{-h}, rs) = ([r^t, rs], 1, 1) = (r^{2t}, 1, 1).$$

Finally, if t is odd and h is odd, then we get

$$\overline{x}^t \overline{y}^h \overline{x}^{-t} \overline{y}^{-h} = (r^t, s, s)(rs, r^h, rs)(r^{-t}, s, s)(rs, r^{-h}, rs)$$

$$= ([r^t, rs], [s, r^h], [s, rs]) = (r^{2t}, r^{-2h}, r^{-2}).$$

Thus, we conclude that $[G_n, G_n]$ is generated by elements of the form

$$\begin{array}{ll} (1, r^{2t}, 1), \qquad (r^{2t}, 1, 1), & t \in 2\mathbb{Z}, \\ (r^{2n_1}, r^{2n_2}, r^2) & n_1, n_2 \in 2\mathbb{Z}+1. \end{array} \tag{3.12}$$

Due to this observation, $(1, 1, r^4) \in [G_n, G_n]$ and hence

$$(1, 1, r^{2t}) \in [G_n, G_n], \qquad \forall\; t \in 2\mathbb{Z}.$$

Moreover, $(r^2, r^2, r^2) \in [G_n, G_n]$.

Since

$$(r^{2t}, 1, 1),\; (1, r^{2t}, 1),\; (1, 1, r^{2t}) \in [G_n, G_n] \qquad \forall\; t \in 2\mathbb{Z},$$

and $(r^2, r^2, r^2) \in [G_n, G_n]$, we conclude that $C_n \subset [G_n, G_n]$.

Since the elements in (3.12) belong to C_n, we also have the inclusion $[G_n, G_n] \subset C_n$.

We proved that $[G_n, G_n]$ indeed consists of elements of the form (3.9). □

Remark 3.5 In Proposition 3.4, it makes sense to consider integers n_1, n_2, n_3 modulo $\mathrm{ord}(r^2)$. Moreover, it makes sense to impose the condition

$$(n_1, n_2, n_3) \in (2\mathbb{Z})^3 \;\text{ or }\; (n_1, n_2, n_3) \in (2\mathbb{Z}+1)^3$$

only in the case when $4 \mid n$. If $n \in 4\mathbb{Z}+2$ or if n is odd, then

$$[G_n, G_n] = \langle\, r^2 \,\rangle \times \langle\, r^2 \,\rangle \times \langle\, r^2 \,\rangle. \tag{3.13}$$

Indeed, if $n = 4t+2$, then $\mathrm{ord}(r^2) = 2t+1$. Hence $\langle\, r^4 \,\rangle = \langle\, r^2 \,\rangle$ and identity (3.13) follows from the inclusion

$$\langle\, r^4 \,\rangle \times \langle\, r^4 \,\rangle \times \langle\, r^4 \,\rangle \subset [G_n, G_n].$$

If n is odd, then the proof of identity (3.13) is easier and we leave it to the reader.

4 The description of $\mathsf{GT}(\mathsf{K}^{(n)})$

Note that

$$\theta(\mathsf{K}^{(n)}) = \mathsf{K}^{(n)}, \qquad \tau(\mathsf{K}^{(n)}) = \mathsf{K}^{(n)}.$$

This follows from the normality of $\mathsf{K}^{(n)}$ in B_3 and the fact that $c \in \mathsf{K}^{(n)}$. In other words, the subgroup $\langle\, \theta, \tau \,\rangle \le \mathrm{Aut}(\mathsf{F}_2)$, preserves $\mathsf{K}^{(n)}$.

Hence the subgroup $\langle\, \theta, \tau \,\rangle \le \mathrm{Aut}(\mathsf{F}_2)$ naturally acts on G_n and $[G_n, G_n]$.

We also have

$$\theta(\overline{z}) = \big(\theta(\overline{xy})\big)^{-1} = \big(\overline{yx}\big)^{-1} = \big((rs, r, rs)(r, s, s)\big)^{-1} = (s, rs, r)^{-1} = (s, rs, r^{-1}).$$

Hence

$$\theta(\overline{z}^2) = \overline{z}^{-2}. \tag{4.1}$$

Let $n_1, n_2, n_3 \in \{0, 1, \dots, \mathrm{ord}(r^2) - 1\}$. Combining (4.1) with $\theta(\overline{x}) = \overline{y}$ and $\theta(\overline{y}) = \overline{x}$, we conclude that, for every $g := (r^{2n_1}, r^{2n_2}, r^{2n_3}) \in [G_n, G_n]$, we have

$$\theta(r^{2n_1}, r^{2n_2}, r^{2n_3}) = (r^{2n_2}, r^{2n_1}, r^{-2n_3}). \tag{4.2}$$

Moreover, since $\tau(x) := y$, $\tau(y) := z$ and $\tau(z) = x$, we have

$$\tau(r^{2n_1}, r^{2n_2}, r^{2n_3}) = (r^{2n_3}, r^{2n_1}, r^{2n_2}). \tag{4.3}$$

Using (4.2), we see that $g := (r^{2n_1}, r^{2n_2}, r^{2n_3}) \in [G_n, G_n]$ satisfies (3.7) if and only if

$$n_1 + n_2 \equiv 0 \mod \mathrm{ord}(r^2).$$

Let us now consider

$$m \in \{0, 1, \dots, K^{(n)}_{\mathrm{ord}} - 1\}, \qquad \gcd(2m+1, K^{(n)}_{\mathrm{ord}}) = 1$$

and assume that m is odd.

Setting

$$g := (r^{2k}, r^{-2k}, r^{2t}),$$

unfolding the right hand side of (3.8) and using (4.3), we get (recall that m is odd):

$$(r^m, s, s)(r^{-2k}, r^{2t}, r^{2k})(r^2 s, r^{-1}s, r^m)(r^{2t}, r^{2k}, r^{-2k})(rs, r^m, rs)(r^{2k}, r^{-2k}, r^{2t})$$

$$= (r^m, s, s)(r^2 s, r^{-1}s, r^m)(rs, r^m, rs)\,(r^{-2k}, r^{-2t}, r^{-2k})(r^{-2t}, r^{2k}, r^{2k})(r^{2k}, r^{-2k}, r^{2t})$$

$$= (r^{m+1}, r^{m+1}, r^{-m-1})(r^{-2t}, r^{-2t}, r^{2t}) = (r^{m+1-2t}, r^{m+1-2t}, r^{2t-m-1}).$$

Thus a pair

$$\big(m, (r^{2k}, r^{-2k}, r^{2t})\big), \qquad k, t \in 2\mathbb{Z} \ \text{ or } \ k, t \in 2\mathbb{Z}+1$$

satisfies (3.8) if and only if

$$m + 1 \equiv 2t \mod \operatorname{ord}(r^2).$$

Notice that, if $4 \mid n$, then we have an additional restriction on n_1, n_2, n_3 to have the same parity. This is equivalent to the congruence $k \equiv \frac{m+1}{2} \mod 2$. Thus we arrive at the following statement:

Proposition 4.1 *Let $n \in \mathbb{Z}_{\geq 3}$ and*

$$\mathcal{X}_n := \big\{ m : m \in \{0, 1, \dots, K^{(n)}_{\text{ord}} - 1\} \ \mid \ \gcd(2m+1, K^{(n)}_{\text{ord}}) = 1 \big\},$$

$$\varkappa(m) := \begin{cases} m+1, & \text{if } 2 \nmid m, \\ -m, & \text{if } 2 \mid m. \end{cases} \tag{4.4}$$

Then

$$\mathsf{GT}^{\heartsuit}_{pr}(\mathsf{K}^{(n)}) = \begin{cases} \big\{ (m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \mid m \in \mathcal{X}_n, \ k \equiv \frac{\varkappa(m)}{2} \text{ mod } 2 \big\} & \text{if } 4 \mid n, \\ \big\{ (m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \mid m \in \mathcal{X}_n, \big\} & \text{if } 4 \nmid n. \end{cases}$$

Proof. For odd m, the desired statement is proved right above the proposition. The case when m is even is easier and we leave it to the reader. □

The following lemma plays an important role in describing the connected component of $\mathsf{K}^{(n)}$ in the groupoid GTSh:

Lemma 4.2 *For every $(m, g) \in \mathsf{GT}^{\heartsuit}_{pr}(\mathsf{K}^{(n)})$,*

$$\ker(T^{\mathrm{PB}_3}_{m,g}) = \mathsf{K}^{(n)} \tag{4.5}$$

and the homomorphism $T^{\mathrm{PB}_3}_{m,g}$ is surjective.

Proof. Since $\psi_n(c) = (1,1,1)$, we have $\mathrm{PB}_3/\mathsf{K}^{(n)} \cong \mathsf{F}_2/\mathsf{K}^{(n)}_{\mathsf{F}_2}$. Hence we may also identify the quotient group $\mathrm{PB}_3/\mathsf{K}^{(n)}$ with the subgroup $G_n \leq D_n^3$ generated by (r, s, s) and (rs, r, rs).

Consider the faithful action of D_n on $\mathbb{Z}/n\mathbb{Z}$: $r(\overline{j}) = \overline{j} + \overline{1}$, $s(\overline{j}) = -\overline{j}$, which defines an injection $D_n \to S_n$.

Let us prove that, for every $(m, g) \in \mathsf{GT}^{\heartsuit}_{pr}(\mathsf{K}^{(n)})$, there exists a triple $(h_1, h_2, h_3) \in S_n^3$ (depending on (m, g)) such that

$$\overline{x}^{2m+1} = (h_1, h_2, h_3)\, \overline{x}\, (h_1^{-1}, h_2^{-1}, h_3^{-1}), \quad g^{-1} \overline{y}^{2m+1} g = (h_1, h_2, h_3)\, \overline{y}\, (h_1^{-1}, h_2^{-1}, h_3^{-1}). \tag{4.6}$$

A direct calculation shows that

$$\overline{x}^{2m+1} = (r^{2m+1}, s, s), \qquad g^{-1} \overline{y}^{2m+1} g = (r^{1-4k} s, r^{2m+1}, r^{1-2\varkappa(m)} s),$$

where $g = (r^{2k}, r^{-2k}, r^{\varkappa(m)})$.

Consider the bijection $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ that sends $\overline{j}$ to $(2\overline{m} + 1) \cdot \overline{j}$ and let b be the corresponding element of S_n.

Setting $h_1 := r^{-2k-m}b$, $h_2 := b$ and

$$h_3 := \begin{cases} b & \text{if } m \text{ is even,} \\ bs & \text{if } m \text{ is odd,} \end{cases}$$

we get a triple of permutations (h_1, h_2, h_3) for which (4.6) holds.

Since $\psi_n(c) = (1,1,1)$, identity (4.6) implies that, for each charming GT-pair (m, g) with the target $\mathsf{K}^{(n)}$, there exists an inner automorphism δ (depending on (m, g)) of S_n^3 such that

$$T_{m,g}^{\mathrm{PB}_3} = \delta \circ \psi_n \,.$$

This implies the desired equality $\ker(T_{m,g}^{\mathrm{PB}_3}) = \mathsf{K}^{(n)}$.

Combining (4.5) with the isomorphism theorem, we conclude that, for every $(m, g) \in \mathsf{GT}_{pr}^{\heartsuit}(\mathsf{K}^{(n)})$, the order of the subgroup $T_{m,g}^{\mathrm{PB}_3}(\mathrm{PB}_3) \le \mathrm{PB}_3/\mathsf{K}^{(n)}$ coincides with the order of the quotient group $\mathrm{PB}_3/\mathsf{K}^{(n)}$.

Thus the homomorphism $T_{m,g}^{\mathrm{PB}_3} : \mathrm{PB}_3 \to \mathrm{PB}_3/\mathsf{K}^{(n)}$ is surjective and the lemma is proved. □

Combining Proposition 4.1 with Lemma 4.2, we get an explicit description of the set $\mathsf{GT}(\mathsf{K}^{(n)})$:

Theorem 4.3 *For every $n \ge 3$, the set of GT-shadows with the target $\mathsf{K}^{(n)}$ is*

$$\mathsf{GT}(\mathsf{K}^{(n)}) = \begin{cases} \left\{(m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \mid m \in \mathcal{X}_n, \quad k \equiv \frac{\varkappa(m)}{2} \ \textit{mod 2}\right\} & \textit{if } 4 \mid n, \\ \left\{(m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \mid m \in \mathcal{X}_n, \right\} & \textit{if } 4 \nmid n, \end{cases} \tag{4.7}$$

where

$$\mathcal{X}_n := \left\{m : m \in \{0, 1, \dots, K_{\mathrm{ord}}^{(n)} - 1\} \mid \gcd(2m+1, K_{\mathrm{ord}}^{(n)}) = 1\right\}$$

and the function $\varkappa$ is defined in (4.4). Furthermore, $\mathsf{K}^{(n)}$ is an isolated object of the groupoid GTSh.

Proof. Due to the second statement of Lemma 4.2, the second condition of Proposition 2.9 is satisfied for every element of $\mathsf{GT}_{pr}^{\heartsuit}(\mathsf{K}^{(n)})$. Thus every charming GT-pair (with the target $\mathsf{K}^{(n)}$) is indeed a GT-shadow, i.e. $\mathsf{GT}(\mathsf{K}^{(n)}) = \mathsf{GT}_{pr}^{\heartsuit}(\mathsf{K}^{(n)})$. Hence Proposition 4.1 implies the first statement of the theorem.

The first statement of Lemma 4.2 implies that, if the target of a GT-shadow is $\mathsf{K}^{(n)}$, then its source is also $\mathsf{K}^{(n)}$. Thus $\mathsf{K}^{(n)}$ is indeed the only object of its connected component in GTSh. □

Theorem 4.3 implies that GT-shadows with the target $\mathsf{K}^{(n)}$ form a (finite) group.

We can now start proving the main surjectivity relation with this intermediate result:

Proposition 4.4 *Let $n, q \in \mathbb{Z}_{\ge 3}$ and $n \mid q$. Then $\mathsf{K}^{(q)} \le \mathsf{K}^{(n)}$ and the reduction map*

$$\mathcal{R}_{\mathsf{K}^{(q)},\mathsf{K}^{(n)}} : \mathsf{GT}(\mathsf{K}^{(q)}) \to \mathsf{GT}(\mathsf{K}^{(n)})$$

is surjective.

Proof. It is sufficient to prove surjectivity in the case when $q = np$, where p is prime. Let $(m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \in \mathsf{GT}(\mathsf{K}^{(n)})$. It is enough to show that there exists such $z \in \mathbb{Z}$ that

- $k \equiv \frac{\varkappa(m+zn)}{2} \bmod 2$,
- $\gcd(2(m+zn)+1, q) = 1$.

Then the pair $(m+zn, (r^{2k}, r^{-2k}, r^{\varkappa(m+nz)})) \in \mathsf{GT}(\mathsf{K}^{(q)})$ gets sent to the pair

$$(m, (r^{2k}, r^{-2k}, r^{\varkappa(m)})) \in \mathsf{GT}(\mathsf{K}^{(n)}).$$

Put $z = 4z_1$, then $m+zn = m+4z_1n \equiv m \bmod 4$, therefore by definition of $\varkappa$: $\frac{\varkappa(m+zn)}{2} \equiv k \bmod 2$. So we are left with $\gcd(2(m+zn)+1, q) = \gcd(2m+8z_1n+1, pn)$. Notice that $2m+1$ is coprime with n, therefore $\gcd(2m+1+8z_1n, n) = 1$. So we need to find such $z_1 : \gcd(2m+8z_1n+1, p) = 1$. In order to do that we need to consider 2 cases:

Case 1. $p \mid n$. Then the statement is obvious because component $2m+8z_1n+1$ is coprime with n and so with p.
Case 2.1. $p \nmid n, p = 2$. Then $2m+8z_1n+1$ is simply odd.
Case 2.2. $p \nmid n, p > 2$. We have $\gcd(p, 8n) = 1$, hence using Chinese Remainder Theorem we can find such $z_1 : p \nmid (2m+8z_1n+1)$. □

It turns out that Proposition 4.4 can be extended to the case when $\mathsf{K}^{(q)} \leq \mathsf{K}^{(n)}$, while $n \nmid q$. To achieve this goal, we will start with the following auxiliary statement:

Proposition 4.5 *For every odd integer $n \geq 3$, we have $\mathsf{K}^{(n)} = \mathsf{K}^{(2n)}$.*

Proof. Due to Remark 3.3, $\mathsf{K}^{(2n)} \subset \mathsf{K}^{(n)}$.

As above, it is convenient to identify the quotient group $\mathsf{F}_2/\mathsf{K}^{(q)}_{\mathsf{F}_2}$ with the subgroup G_q of D_q^3 generated by $\overline{x}$ and $\overline{y}$.

Let $J_q := \langle r^2 \rangle \times \langle r^2 \rangle \times \langle r^2 \rangle \leq G_q$. It is easy to see that

$$|J_q| = \begin{cases} q^3, & \text{if } q \text{ is odd,} \\ \left(\frac{q}{2}\right)^3, & \text{if } q \text{ is even} \end{cases}$$

and $G_q/J_q \cong C_2 \times C_2$ (there are 4 cosets: $\{J_q, \overline{x}J_q, \overline{y}J_q, \overline{xy}J_q\}$ and the cosets $\overline{x}J_q, \overline{y}J_q, \overline{xy}J_q$ have order 2 in G_q/J_q). Therefore, $|\mathrm{PB}_3 : \mathsf{K}^{(q)}| = |\mathsf{F}_2/\mathsf{K}^{(q)}_{\mathsf{F}_2}| = 4|J_q|$. Thus,

$$|\mathrm{PB}_3 : \mathsf{K}^{(n)}| = \begin{cases} 4q^3, & \text{if } q \text{ is odd,} \\ 4\left(\frac{q}{2}\right)^3, & \text{if } q \text{ is even.} \end{cases}$$

Therefore, for an odd integer n, we have $|\mathrm{PB}_3 : \mathsf{K}^{(n)}| = 4n^3 = 4\left(\frac{2n}{2}\right)^3 = |\mathrm{PB}_3 : \mathsf{K}^{(2n)}|$. Hence $|\mathsf{K}^{(n)} : \mathsf{K}^{(2n)}| = 1$ and we have the desired equality $\mathsf{K}^{(2n)} = \mathsf{K}^{(n)}$. □

Proposition 4.6 *Let $n, q \geq 3$. Then*

$$\mathsf{K}^{(q)} \subset \mathsf{K}^{(n)} \iff n \mid K^{(q)}_{\mathrm{ord}}.$$

Proof.

Case 1: $\Leftarrow$
If $n \mid K^{(q)}_{\mathrm{ord}}$, then $n \mid \mathrm{lcm}(q, 2)$. If q is even, then we have $n \mid q$, and the desired set inclusion follows from Remark 3.3. If q is odd, then $n \mid 2q$ and we obtain $\mathsf{K}^{(q)} = \mathsf{K}^{(2q)} \subset \mathsf{K}^{(n)}$ from the

same remark.

Case 2: $\Rightarrow$
Note that $x_{12}^{K^{(q)}_{\mathrm{ord}}} \in \mathsf{K}^{(q)} \subset \mathsf{K}^{(n)}$. Then $(1,1,1) = \psi_n(x_{12}^{K^{(q)}_{\mathrm{ord}}}) = (r^{K^{(q)}_{\mathrm{ord}}}, 1, 1)$, and hence $n \mid K^{(q)}_{\mathrm{ord}}$. □

Finally, we are ready to prove the stronger version of Proposition 4.4:

Theorem 4.7 *Let $n, q \in \mathbb{Z}_{\geq 3}$ with $\mathsf{K}^{(q)} \subset \mathsf{K}^{(n)}$. Then the reduction map*

$$\mathcal{R}_{\mathsf{K}^{(q)},\mathsf{K}^{(n)}} : \mathsf{GT}(\mathsf{K}^{(q)}) \to \mathsf{GT}(\mathsf{K}^{(n)})$$

is surjective.

Proof. According to Proposition 4.6, $n \mid K^{(q)}_{\mathrm{ord}}$. If q is even, we have $n \mid q$ and the desired surjectivity follows from Proposition 4.4. If q is odd, we have $n \mid 2q$. Therefore, the map $\mathsf{GT}(\mathsf{K}^{(2q)}) \to \mathsf{GT}(\mathsf{K}^{(n)})$ is surjective. Then the desired statement follows from the fact that $\mathsf{GT}(\mathsf{K}^{(q)}) = \mathsf{GT}(\mathsf{K}^{(2q)})$. □

References

[1] G.V. Belyi, Galois extensions of a maximal cyclotomic field, Izv. Akad. Nauk SSSR Ser. Mat. **43**, 2 (1979) 267–276.

[2] V.A. Dolgushev and J.J. Guynee, GT-shadows for the gentle version of the Grothendieck-Teichmueller group, in preparation.

[3] V. A. Dolgushev, K. Q. Le and A. A. Lorenz, What are GT-shadows? to appear in Algebr. Geom. Topol., arXiv:2008.00066

[4] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with Gal($\overline{\mathbb{Q}}/\mathbb{Q}$), Algebra i Analiz **2**, 4 (1990) 149–181

[5] P. Guillot, The Grothendieck-Teichmueller group of a finite group and G-dessins d'enfants. *Symmetries in graphs, maps, and polytopes,* 159–191, Springer Proc. Math. Stat., **159**, Springer, 2016; arXiv:1407.3112.

[6] P. Guillot, The Grothendieck-Teichmueller group of $PSL(2,q)$, arXiv:1604.04415.

[7] D. Harbater and L. Schneps, Fundamental groups of moduli and the Grothendieck-Teichmueller group, Trans. of the AMS **352**, 7 (2000) `https://webusers.imj-prg.fr/~leila.schneps/HS.pdf`

[8] Y. Ihara, On the embedding of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) into GT, London Math. Soc. Lecture Note Ser., **200** Cambridge University Press, Cambridge, 1994, 289–321

[9] C. Kassel and V. Turaev, Braid groups, *with the graphical assistance of Olivier Dodane,* Graduate Texts in Mathematics, **247.** Springer, New York, 2008. xii+340 pp.

[10] P. Lochak and L. Schneps, Open problems in Grothendieck-Teichmueller theory, *Problems on mapping class groups and related topics,* 165–186. Proc. Sympos. Pure Math., **74** AMS, Providence, RI, 2006

[11] F. Pop, Little survey on I/OM and its Variants and their relation to (variants of) GT, Topology and its Applications **313**, Paper No. 107993

[12] T. Szamuely, Galois groups and fundamental groups, *Cambridge Studies in Advanced Mathematics,* **117**. Cambridge University Press, Cambridge, 2009.

[13] J. Xia, GT-Shadows related to finite quotients of the full modular group, Master Thesis, 2021, Temple University, https://math.temple.edu/~vald/JingfengXiaMasterThesis.pdf