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PARTIAL ORDERINGS OF MINORS IN THE POSITIVE GRASSMANNIAN

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ABSTRACT. In this paper, we research the partial order of minors of the positive Grassmannian (i.e. a space that parametrizes $k \times n$ matrices with all $k \times k$ minors positive) with a fixed maximal set of largest minors. This is connected to a combinatorial structure called a circuit graph. The maximal set of the largest minors corresponds to some cycle of length n in the circuit graph, and then, abstractly, the further away in the graph some minor is from maximal the less its value. But this further is some non-trivial property called cubical distance, which be explained in the paper.

1. INTRODUCTION

Since its introduction by Postnikov in 2006 [6], the totally positive (TP) Grassmannian has been studied extensively, unveiling a rich combinatorial structure as well as numerous applications to physics and beyond. Due to its manifold connections to other combinatorial objects, as well as connections to objects important to physics such as the amplituhedron [1] and to integrable systems [4], it is very natural to study in detail the structure of the totally positive Grassmannian itself, in hopes that this will lead to insights in related fields. One such direction is to study equalities and inequalities between minors in the TP Grassmannian. Inequalities between products of minors have been studied as early as 2004, by Skadera in [8] and others ([5] [7]).

It is also of interest to study inequalities between the minors themselves. This was first done by Farber and Postnikov in [3], and then extended by Farber and Mandelshtam in [2]. In particular, arrangements of largest equal minors were shown in [3] to be in bijection with the simplices of a triangulation of the hypersimplex studied by Stanley, Sturmfels, Lam, and Postnikov. In [2], second largest minors were shown to correspond to the facets of the simplices of the triangulation. For the t^{th} largest minors, [2] conjectures that these are related to points of so-called cubical distance t on the dual graph of the triangulation. In this paper, we address this conjecture and discuss steps that we have taken towards proving it.

This paper is structured as follows. In section 2, we give relevant background and definitions to understand the rest of the paper. In section 3, we are introducing a combinatorial connection between an arrangement of minors and the circuit graph, introducing the cubical distance on the circuit graph and conjecture 2.10 what we intended to prove. Section 4 is dedicated to our own research results on trying to simplify a cubical distance concept by finding an alternative approach that we called the stratification of the circuit graph, we introduced some conjectures (see 4.9 and 4.7)about this structure that is enough to prove conjecture 4. In section 5 we wrote some small things about not partial but absolute order.

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2. Preliminaries

Definition 2.1. let V be an $n-dimensional$ real vector space, and let k be an integer such that $0 \leq k \leq n$. The *Grassmannian* $Gr(k, V)$ is the space of all $k-dimensional$ subspaces of V.

The space $G(k, R^n)$ is denoted by $Gr(k, n)$. For $A \in Gr(k, n)$ let A_I, for $I \in \binom{[n]}{k}$, denote its $k \times k$ submatrix with column set I and $\Delta_I = \det(A_I)$. The Δ_I are called Plücker coordinates and give an embedding of $Gr(k, n)$ into $\mathbb{P}^{\binom{n}{k}-1}$, the $\binom{n}{k}$-dimensional projective space. Further we will be mostly interested in positive Grassmannian, $Gr^+(k, n)$. It is the subset of $G(k, n)$ such that $\det(A_I) \geq 0$ holds for all $A \in Gr^+(k, n)$ and $I \in \binom{[n]}{k}$.

Definition 2.2. Let $\mathbb{U} = (\mathbb{U}_0, \mathbb{U}_1, \cdots \mathbb{U}_l)$ be an ordered set-partition of $\binom{[n]}{k}$ according to Plücker coordinates in $A \in Gr^+(k, n)$ such that:
(1) $\Delta_I = 0$ for $I \in \mathbb{U}_0$
(2) $\Delta_I = \Delta_J$ if I,$J \in \mathbb{U}_i$
(3) $\Delta_I < \Delta_J$ if $I \in \mathbb{U}_i$ $J \in \mathbb{U}_j$ with $i < j$
We call such an ordered set partition an arrangement of minors.

This partition cannot be arbitrary and has an interesting combinatorial structure, which is the focus of this paper.

Definition 2.3. We say that A_I, $I \in \binom{[n]}{k}$ is a t-largest minor in $A \in Gr^+(k, n)$ if there exists an arrangement of minors $\mathbb{U}$ of $G(k, n)$ such that $I \in \mathbb{U}_{l-t}$.

The next definitions are a crucial combinatorial feature of arrangements of minors

Definition 2.4. For a multiset S of elements from [n], let $Sort(S)$ be the non-decreasing sequence obtained by ordering the elements of S. Let $I, J \subset \binom{[n]}{k}$ and let $Sort(I \cup J)=(a_1; a_2; \cdots ; a_{2k})$. Define:

$$Sort_1(I, J) := \{a_1; a_3; \cdots ; a_{2k-1}\}, \quad Sort_2(I, J) := \{a_2; a_4; \cdots ; a_{2k}\}.$$

A pair I;J is called sorted if $Sort_1(I, J) = I$ and $Sort_2(I, J) = J$, or vice versa

For example the pair {1,2,5},{3,4,6} is not sorted and $Sort_1(I, J) =${1,2,3} , $Sort_2(I, J) =${4,5,6} which means that the pair {1,2,3},{4,5,6} is sorted. The next inequality is known as Skandera inequality and it is the main tool for building the partial order in our partition.

Theorem 2.5 ([8]). *Let $I, J \in \binom{[n]}{k}$ be a pair which is not sorted. Then $\Delta_{sort_1(I,J)}\Delta_{sort_2(I,J)} \geq \Delta_I \Delta_J$.*

Definition 2.6. A collection $I = I_1, I_2, \cdots , I_r$ of elements in $\binom{[n]}{k}$ is called sorted if I_i, I_j are sorted, for any pair $1 \leq i < j \leq n$. Equivalently, if $I_i = \{a^i_1 < a^i_2 < \cdots < a^i_k\}$ for all i then I is sorted if (after possible reordering of the I_i's) we have

$$a^1_1 \leq a^2_1 \cdots \leq a^r_1 \leq a^1_2 \leq a^2_2 \leq \cdots \leq a^r_k$$

For $I \in \binom{[n]}{k}$ let ϵ_I be the vector $\{\epsilon_1, \epsilon_2, \cdots , \epsilon_n\} \in \{1, 0\}^n$ where $\epsilon_i = 1$ if $i \in I$ otherwise $\epsilon_i = 0$. We will use such designation later to define circuit.

As was shown in [3] the arrangement of largest minors in a matrix is necessarily a sorted set, and every possible maximal sorted set can be an arrangement of largest minors. Any maximal (by inclusion) sorted set of $\binom{[n]}{k}$ contains n elements.

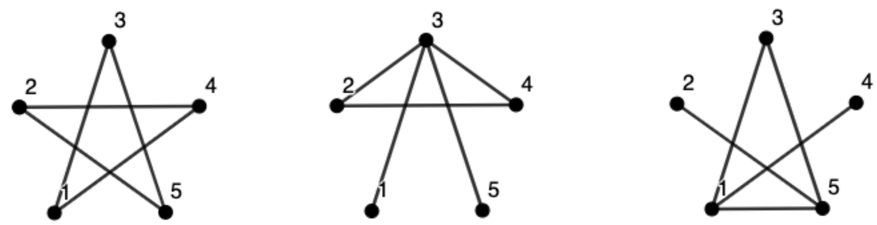

Figure 1. An example of all maximal thrackles for $\binom{[5]}{2}$ up to rotation and symmetry.

For $k = 2$ a sorted set can be represented on a graph where vertices form a regular n-gon and every two edges intersect or have a common vertex. This kind of graph is called a thrackle. See Figure 1 for some examples.

Now we want to define the graph $\Gamma_{k,n}$, which is a dual graph of a certain triangulation of the hypersimplex, and as we will show later there is a connection between some features of the dual graph and the partial order on the partition of minors induced by a maximal arrangement.

Definition 2.7. The Dual Graph $\Gamma_{k,n}$ is the graph whose vertices are all maximal sorted sets and two vertices I, J are connected if $|I \cap J| = n - 1$, in the language of thrackle, it can be explained that two vertices (thrackles) are connected if we can obtain one thrackle from the other by switching one edge.

Figure 3 shows some dual graphs. For us, it is important that every vertex of a graph corresponds to a maximal arrangement of minors, and conceptually, there is a correlation between how big, the determinant of the matrix can be, and how close the first vertex of the dual graph where this Δ_I appears, to vertex that corresponds to maximal minors.

But to define this "closeness" we have to define the so-called cubical distance. To give intuition on cubical distance let us consider the blue edges in Figure 2, and note that they form a square, while the red edges form a 3-dimensional cube. We say that two vertices J_1, J_2 in $\Gamma_{(k,n)}$ are of cubical distance 1 if both of them lie on a certain cube (of any dimension). For example, vertices a and b from Figure 2 are of cubical distance 1 since both of them lie on a 1-dimensional cube (which is just an edge). similarly, a and c are of cubical distance 1 (both of them lie on a square), as well as c and d (both of them lie on a 3-dimensional cube).

Definition 2.8. Let $J_1, J_2 \subset \binom{[n]}{k}$ be maximal sorted collections, then we say that the cubical distance $cube_d(J_1, J_2) = D$ if we can arrive from J_1 to J_2 by moving along D cubes in $\Gamma_{(k,n)}$ and D is minimal with regard to this property. Also for $W \in \binom{[n]}{k}$ and $J_1 \subset \binom{[n]}{k}$ we say that $cube_d(W, J_1) = cube_d(J_1, W) = D$, if for every J_2 that contains W, $cube_d(J_1, J_2) \geq D$ and there exist such J_2 that this inequality turns into equality.

Definition 2.9. Let $J \subset \binom{[n]}{k}$ be arrangment of largest minors, then we say that $W \in \binom{[n]}{k}$ is $(\geq t, J)$-largest minor if for any arrangement $\mathbb{U} = (\mathbb{U}_0, \mathbb{U}_1, \cdots \mathbb{U}_l)$ where $\mathbb{U}_l = J$ the following holds, $W \notin \mathbb{U}_{l-t+1}, \mathbb{U}_1, \cdots \mathbb{U}_l$.

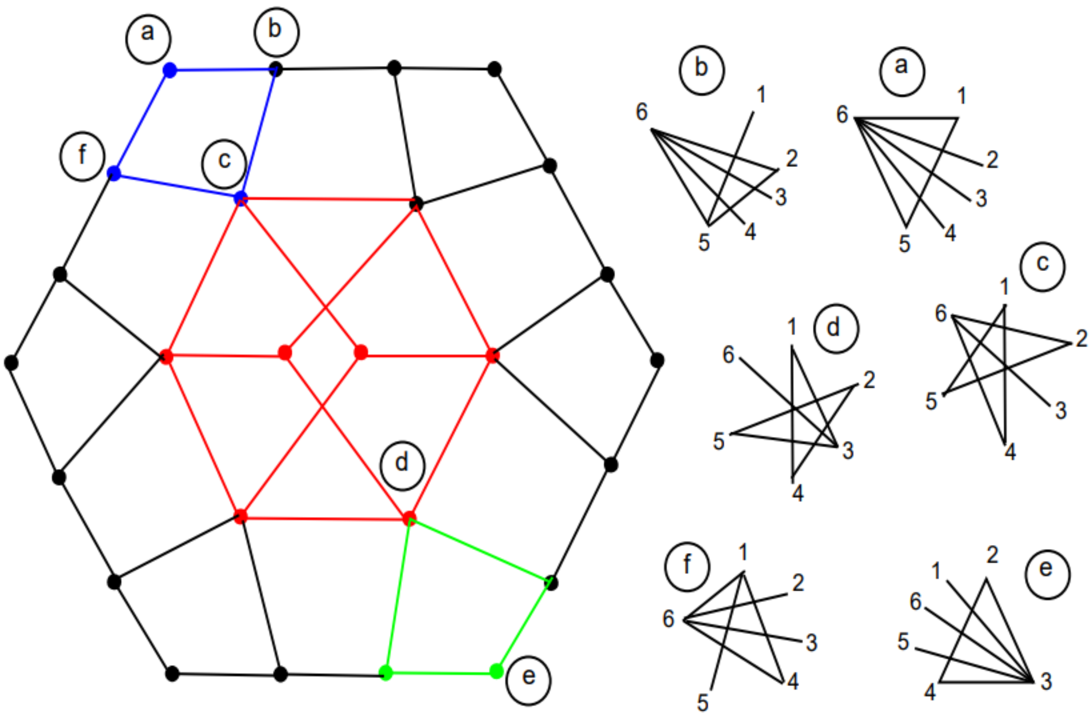

Figure 2. $\Gamma_{2,6}$ where some vertices attached to corresponding thrackle

And now we are ready to present the conjecture from [2] that we are working on.

Conjecture 2.10. *Let $J \subset \binom{[n]}{k}$ be an arrangement of largest minors. If $cube_d(W, J) = t$ then W is a $(\geq t, J)$-largest minor.*

In the following part of our paper we will introduce some other ways to simplify and understand the concept of cubical distance, since in present form it is structure is difficult to work with.

3. Dual Graph and Circuit triangulation

Here we will demonstrate an alternative approach to maximal sorted sets.

Definition 3.1. We define $G_{n,k}$ to be the directed graph whose vertices are $\{\epsilon_I\}_{I \in \binom{[n]}{k}}$ and two vertices $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_n)$ and ϵ' are connected by an edge oriented from ϵ to ϵ' if there exists some $i \in [n]$ such that $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ and the vector ϵ' is obtained by ϵ by switching $\epsilon_i, \epsilon_{i+1}$(note that we are working *mod* n so if $i = n$ then $i + 1 = 1$. We call the graph $G_{n,k}$ the **circuit** graph.

A circuit in $G_{k,n}$ of minimal length must be of length n, for an example see Figure 4. And the most interesting fact about the minimal circuits is that they exactly match the maximal sorted sets, so every minimal circuit is some maximal sorted set and vice versa. From now on we will think of maximal sorted sets as minimal circuits.

Now we introduce the connection between cubical distance and circuit triangulation that was described in [2].

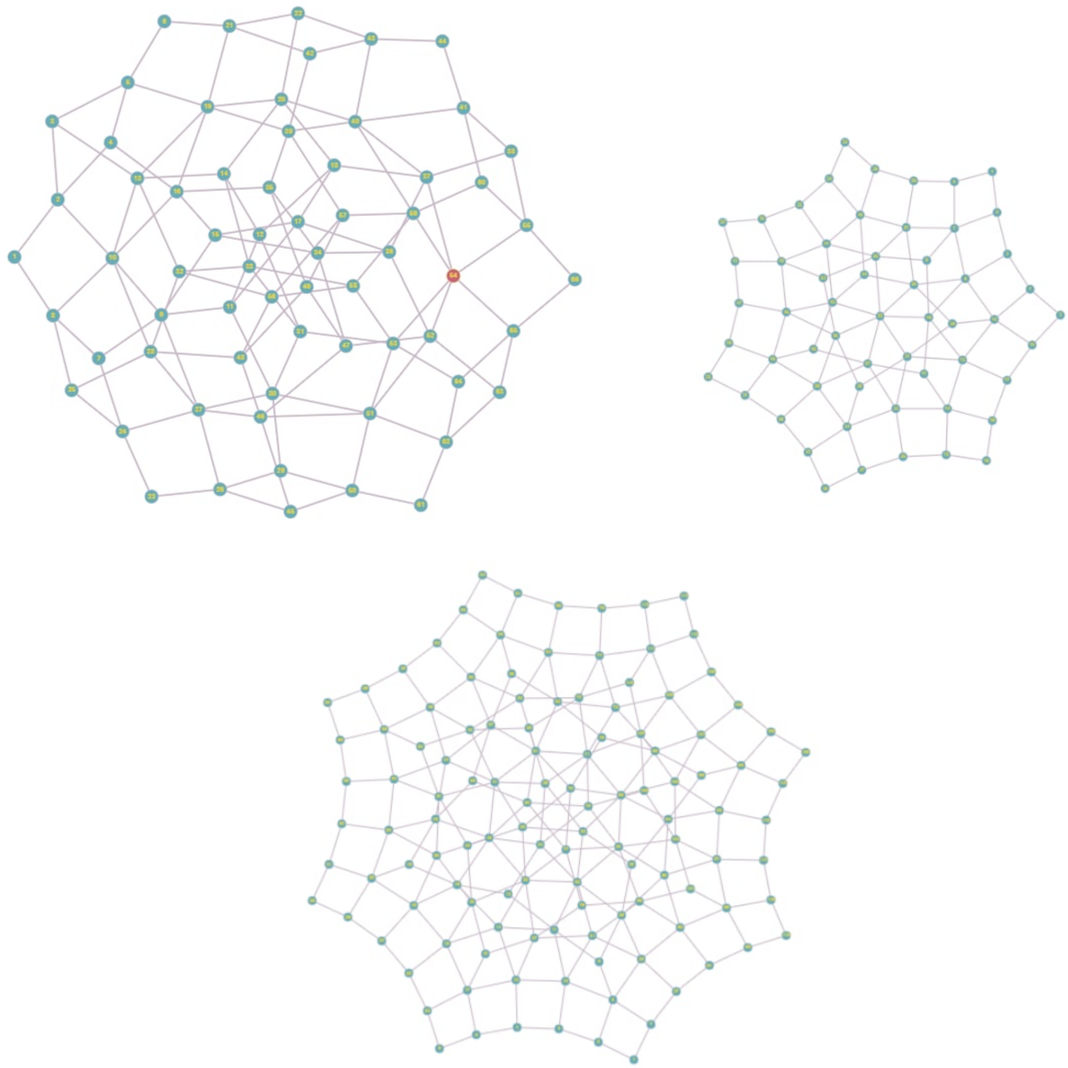

Figure 3. $\Gamma_{3,6}$ (left-top),$\Gamma_{2,7}$ (right-top),$\Gamma_{2,8}$ (at the bottom)

Claim 3.2. *All vertices of a t-dimensional cube correspond to all permutations of some t non-intercepting detours.*

For example, a circuit with all the following detours in Figure 4 forms a 3-dimensional cube in $\Gamma_{3,6}$

Proof. If some circuit has t non-intersecting detours then these t edges of the dual graph coming from this vertex will be completed to a cube because any subset of these detours can be applied and these will be exactly the 2^t vertices of our cube. ■

From this fact flow plenty of interesting facts about the dual graph, for example, that the maximal dimension of any cube in the graph is $\lceil \frac{n}{2} \rceil$. And more importantly, now we can interpret cubical distance as follows:

Claim 3.3. *The cubical distance equals t if and only if C_i is obtained from C_j (where C_i C_j are minimal circuits) by a series of t actions like in 3.2 See Figure 6 for a picture of*

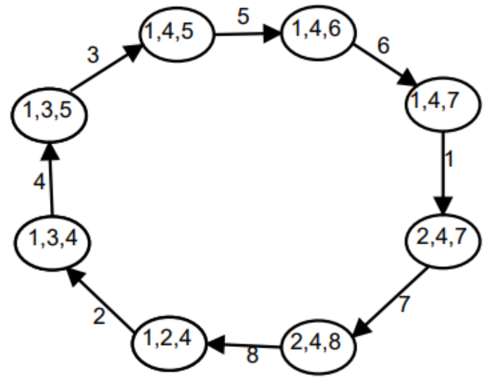

Figure 4. This is some minimal circuit in $G_{3,6}$, and one can easily check that all elements are sorted

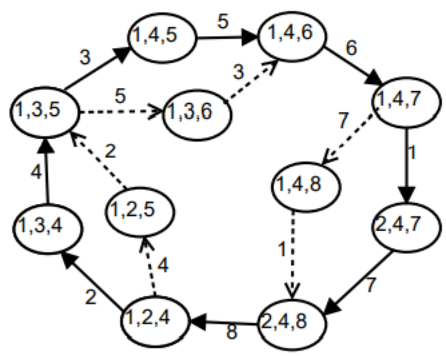

Figure 5. Here we can change $\{1,4,5\}$ to $\{1,3,6\}$, $\{2,4,7\}$ to $\{1,4,8\}$ and $\{1,3,4\}$ to $\{1,2,5\}$.And we can apply any subset of these detours, so this will be 8 vertices of some 3-dimensional cube in $\Gamma_{3,6}$.

every possible set of detours for certain triangulation, each different color corresponds to the element at a certain cubical distance:

4. Stratification of circuit graph

In this section, we shall introduce the alternative approach to the concept of cubical distance. We will define this stratification recursively.

Definition 4.1. The Column-Stratification $R = \{R_0, R_1, \cdots R_n\}$ of a circuit graph $G_{n,k}$ is the division of its vertices according its value mod(n). A vertex x=$\{x_1, x_2, \cdots x_n\} \in G_{n,k}$ is in R_t if sum of i where $x_i = 1$ is equal to t mod(n).

Such a stratification is comfortable for us because every edge is directed from R_i to R_{i+1}. This stratification is cyclic meaning that R_{n+i}= R_i. For some vertex v we call incoming and outgoing edges left and right, respectively.

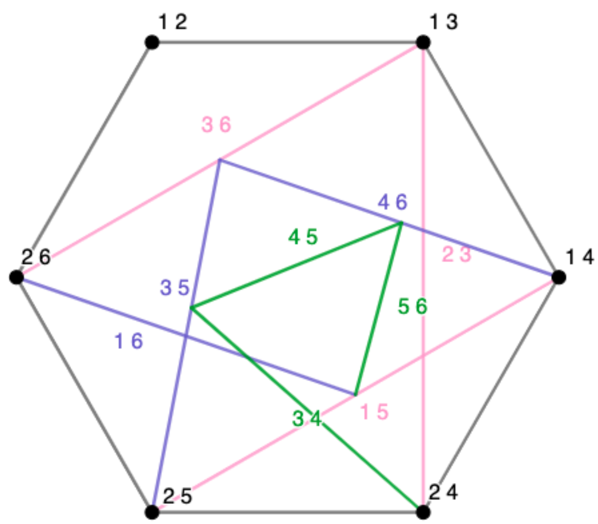

Figure 6. Here the cubical distance from the maximal set (black) to pink vertices is 2, to purple is 3, and to green is 4

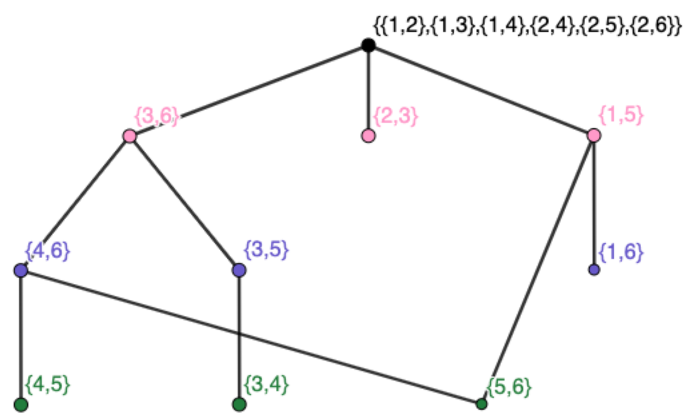

Figure 7. This picture illustrates all inequalities that can be obtained from the particular maximal set (which is black)

Definition 4.2. The row-Stratification $S = \{S_0, S_1, \cdots S_l\}$ of the circuit graph $G_{n,k}$ with maximal circuit $W \subset G_{n,k}$ is defined inductively:
(1) S_0=W
(2) S_{t+1} consists of vertices that were not in any S_i with $i \leq t$ and has two edges to vertices from S_i one in and one out (i.e one left and one right).

The maximal circuit W can be any circuit in $G_{n,k}$. The column-Stratification gives us better picture and understanding of construction while the Row-Stratification is another approach to cubical distance. Assume that we have $A \in Gr^+(k,n)$ such that there exist n maximal minors so that they form a maximal sorted arrangement W which is a circuit in $G_{n,k}$. S_0=W so all minors are on cubical distance 0, S_1 are on distance 1 and so on. The purpose of this stratification is to change the cubical distance to something easier to work with. See figure 8 for an example.

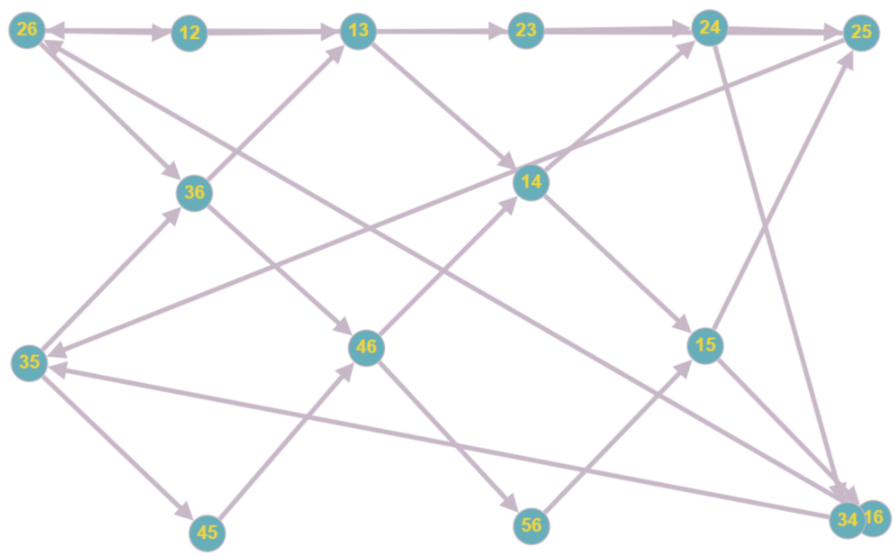

Figure 8. It is a picture that illustrates row and column stratifications of $G_{2,6}$ with maximal circuit $\{\{2,6\};\{1,2\};\{1,3\};\{2,3\};\{2,4\};\{2,5\}\}$.

Theorem 4.3. *The Row-Stratification exists for every circuit graph $G_{n,k}$ and maximal circuit $W \subset G_{n,k}$.*

Proof. The main idea is to use the fact that a dual graph is connected, i.e that we can reach every circuit from every other circuit by a set of detours. Now, let us suppose that at some point in our algorithm, we cannot add any vertex, it means that every vertex that isn't in our S yet has at most one connection to S. let us notice now that if we change the maximal circuit to other maximal circuits where all vertices are from S the situation will remain the same. But we know that we may reach every circuit by a set of detours, but on the other hand, while doing detours, we always remain inside S, so stratification always exists. ■

We would like to prove the equivalence of the stratification we described and the stratification according to cubical distance. For some maximal C_0, let us call C_i the set of minors at cubical distance i.

Proposition 4.4. *If $S_0 = C_0$, then $S_1 = C_1$.*

Proof. Here definitions of C_1 and S_1 actually coincide, both C_1, S_1 are vertices connected twice to the maximal set. ■

Now we will describe what will happen with the stratification after making the detour.

Theorem 4.5. *If S_0' obtained from S_0 by detour, then the level of all members of the circuit graph is less than by 1, so each vertex in S_i or stayed in S_i, or went to $S_{\pm 1+i}$.*

Proof. If we suppose for the sake of contradiction that A moved from S_i to $S_{<i-1}$ then some of its on-top neighbors B on S_{i-1} have to move to at least $S_{<i-2}$, but we can now say the same for B, so that will eventually lead us to that some vertex on S_1 has to be moved by at least two levels on top, what is actually impossible. ■

Here is our result about the relationship between cubical distance and our stratification:

Theorem 4.6. *If $S_0 = C_0$, then any vertex in C_i is on $S_{\leq i}$.*

Proof. The statement easily follows from two previous statements. Suppose that for some $A \in C_i$, also $A \in S_{>i}$, then by definition of being in C_i there is a sequence of i detours after which A will be in C_0. If we will apply this sequence to S_0, then each time level of A in the stratification will decrease on at most 1, so it will be impossible to A get to S_0 after i detours. ■

Actually, we believe that our stratification and division according to cubical distance coincide. We can prove it up to S_4 and again checked it for some large grassmanians using code.

Conjecture 4.7. *If $S_0 = C_0$, then $S_i = C_i$ for all i.*

Now we want to formulate the main conjecture about the structure of the stratification, first, we need a small lemma:

Lemma 4.8. *Every two vertices A, B that are connected with some vertex C by two incoming vertices are connected with some other vertex D by outgoing vertices and vice versa.*

Proof. Let us look at C, in fact, A, B are increasing (or decreasing) two distinct, non-consecutive (because otherwise increasing one of these wouldn't be possible) numbers by 1. So D is just a vertex where these two numbers increased simultaneously. ■

See Figure 9 for an example of such A, B, C, D.

Now we are interested in possible positions of D in our stratification regarding A, B:

Conjecture 4.9. *If A, B belonging to S_i, S_j respectively with $i \neq j$, then D belongs to S_k, where $k \geq min(i, j)$. And if $i = j$, then common vertices lies on $S_{\leq i+1}$*

We checked this using code for some random S_0 up to $Gr(10, 20)$. From this conjecture follows immediately many interesting things, in particular, that there are no two vertices on $S_{>n-1}$ connected to one vertex on S_n cause if we apply conjecture to another common vertex we will get a contradiction. Figure 10 visualizes the allowed location of C, D with green, and not allowed with red (note that there are two cases depending on whether or not $i = j$. Now we see how if this conjecture is true, it can prove some really good results about the arrangements of minors. Note that in all future conjectures and theorems, when we are writing some relation on some minors regarding its position in stratification we mean that this relation holds with some fixed maximal set S_0.

Theorem 4.10. *If A is a vertex in S_n so there exist $B \in S_{<n-1}$ connected to A, such that $\Delta_A < \Delta_B$*

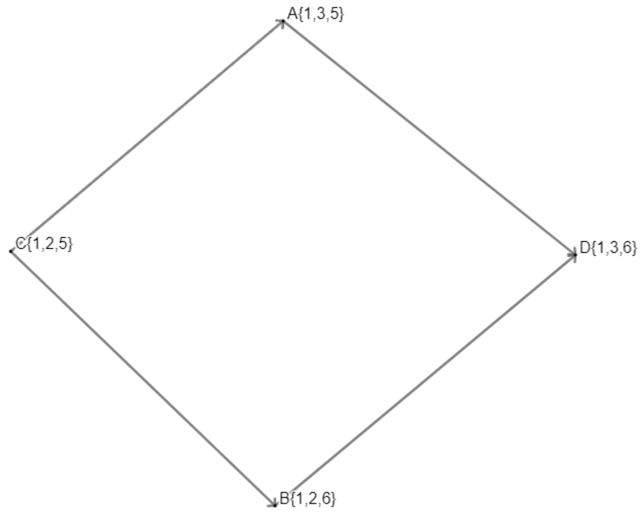

Figure 9

Proof. Assume that edge from A goes to B it means that B is on the right side from A. It is just for our convenience. Let $A \in R_m$ then $B \in R_{m+1}$. Let's call such vertex long because the difference between levels is more than one. From circuit stratification, it follows that there exists such vertex $B_1 \in R_m$ that is higher than B and is connected to it. Hence from lemma 4.8 there exists vertex $A_1 \in R_{m-1}$ that is connected to A and B_1 moreover it is not higher than S_{n-1} from rules of stratification (note that here we do not need the conjecture 4.9, because even if A_1 is lower than A it does not matters). Therefore edge A_1B_1 should be long. And from Scandera inequality:

$$\Delta_A \Delta_{B_1} < \Delta_B \Delta_{A_1}$$

So we see that $\Delta_A < \Delta_B$ equivalent to $\Delta_{A_1} < \Delta_{B_1}$. So we have an algorithm that allows us to obtain a new long edge B_i higher than B_{i-1} and $\Delta_{A_i} < \Delta_{B_i}$ implies $\Delta_{A_{i-1}} < \Delta_{B_{i-1}}$, so for some $B_k \in S_0$, which means that $\Delta_{A_k} < \Delta_{B_k}$ holds, so $\Delta_{A_{k-1}} < \Delta_{B_{k-1}}$ holds, and eventually we will end up with that $\Delta_A < \Delta_B$, what we intended to prove. ∎

We would like to give one more proof of this statement that uses conjecture 4.9:

Proof. We prove it by induction on n. The base is immediate, so now let us assume that it holds for all $S_{<n}$. First, if $B \in S_0$, then it obviously holds for all n, so we might assume that B is not maximal, then by construction of the stratification, there exists C, such that B is connected to C, $C \in S_{<k}$ and C and A are in the same R_p, so by 4.9, there exist such D, that $D \in S_{<n}$, and again, by the construction of the stratification D has to be in S_{n-1} because otherwise A would be connected to incoming and outgoing vertices on a level higher than $n-1$, what is a contradiction to that $A \in S_n$. let us notice that this conjecture holds for C, D, so $\Delta_C > \Delta_D$, and applying Skandera inequality for A, B, C, D, we obtain:

$$\Delta_D \Delta_B > \Delta_A \Delta_C$$

so since we know that $\Delta_C > \Delta_D$ we have $\Delta_B > \Delta_A$, so we are done. ∎

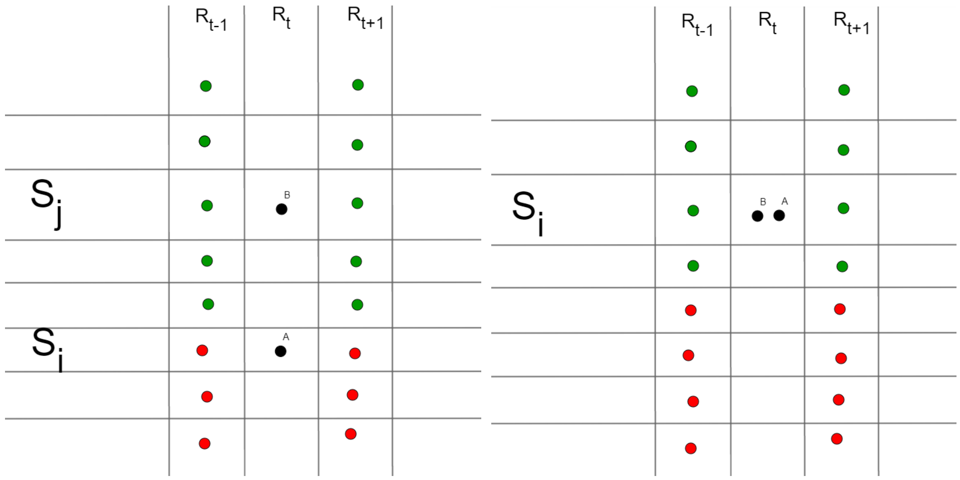

Figure 10

We understood the relations between long edges, but unfortunately, we cannot prove the same for edges of length 1, but the following conjecture gives us a relation for at least one edge of length 1 for any vertex:

Conjecture 4.11. *If A is a vertex in S_n there exists B and B_1 connected to A, incoming and outgoing respectively, such that B and B_1 are higher in stratification and $\Delta_A < \Delta_B$ and $\Delta_A < \Delta_{B_1}$.*

Proof. Here we will use induction on n. And the base is immediate, so now let us consider two cases (see Figure 11). First, if A is connected to $B \in S_{<n+1}$. In such a case, we might take B as in the previous proof, but instead of taking arbitrary C, we will take such C that $\Delta_B < \Delta_C$ (by induction assumption), then we take D and apply the Skandera inequality:

$$\Delta_D \Delta_B > \Delta_A \Delta_C$$

here we know that Δ_C bigger than both Δ_D, Δ_B, so we get $\Delta_D > \Delta_A$,so D and B are such vertices for A. The second case will be almost the same. Again, we take arbitrary B, then take C connected to B and bigger than it and of the proper type, and let us notice that in this case, D can be only on S_{n-1}, so here we will obtain Skandera inequality with $\Delta_C > \Delta_B$, so we will get $\Delta_D > \Delta_A$ so D is desired vertex, then we can do it symmetrically to get D_1 from the other side for B_1 ∎

The last conjecture together with an equivalence between stratification and cubical distance proves Conjecture 2.10 immediately, since for any vertex in S_i we can obtain a chain of length $i+1$ such that on every step the next minor will be bigger.

5. Arrangements of minors

First of all, we suggested the following conjecture:

Conjecture 5.1. *Let $J \subset \binom{[n]}{k}$ be an arrangement of largest minors, then the two following statements if $cube_d(W, J) = t$ then W is (t, J)-largest minor.*

It looks exactly the same as Conjecture 2.10, but instead of $(\geq t, J)$ we have (t, J). We don't know yet how to approach this problem, because the example suggested in [3] for maximal arrangements does not always satisfy this conjecture. Furthermore, we have shown

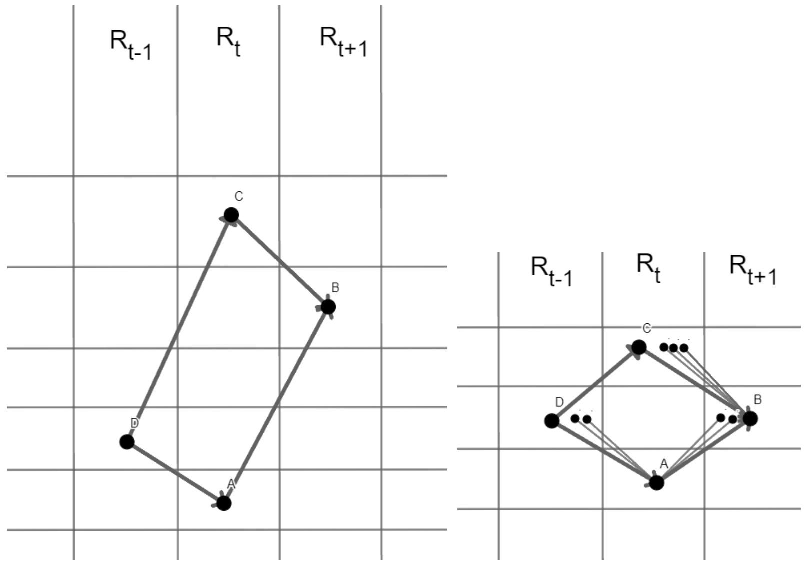

Figure 11

that for some J, it is actually impossible for all $(2, J)$ largest minors to simultaneously be second largest minors. However, individually they all can be. So we likely have to build completely new structures or improve existing ones. Also, it might be easier to prove this conjecture for $k = 2$ or some additional weaker versions of the conjecture. Then we might think of some other properties of the partitions, because for example, when we suggested that there exists a matrix where every minor on cubical distance 2 is really second it appears to be false for $Gr^+(2, 6)$.

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