The Furstenberg property in Puiseux monoids

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Monoids

Notation: Let $\mathbb{N}$ denote the positive integers and $\mathbb{P}$ denote the primes.

Definition (Puiseux Monoid)

A Puiseux monoid is a set $M \subseteq \mathbb{Q}_{\geq 0}$ (the nonnegative rationals) such that

- $a + b \in M$ for all $a, b \in M$.
- $0 \in M$.

Examples

- $\mathbb{Q}_{\geq 0}$ is a Puiseux monoid.
- The set of nonnegative integers $\mathbb{N}_0$ is a Puiseux monoid.
- $\mathbb{N}_0 \setminus \{1\}$ is a Puiseux monoid.

For the rest of this presentation, we abbreviate to just monoid.

Motivation

The study of Puiseux monoids are motivated by:

- Grams used Puiseux monoids to disprove Cohn's conjecture that any atomic domain must satisfy the ACCP.
- Gotti and Zafrullah used Puiseux monoids to argue that the property of having finitely many irreducible divisors does not ascend from a monoid to its monoid algebras.

Generators

We can specify monoids more easily by using generators.

Definition

Suppose that S is any subset of $\mathbb{Q}_{\geq 0}$. It can be shown that the intersection M of all monoids containing S is a monoid itself. The monoid generated by S is defined to be this monoid.

The monoid M is essentially the smallest monoid containing S. If M is the monoid generated by S, it will be denoted by $M = \langle S \rangle$. When we write S in set theory notation, we will omit the curly braces and only use the angle brackets.

Example

The monoid $\mathbb{N}_0$ is the monoid generated by $\{1\}$ and so $\mathbb{N}_0 = \langle 1 \rangle$.

Divisibility in monoids

Let M be a monoid.

Given elements $a, b \in M$, we say b divides a if there exists c in M such that $a = b + c$.

Examples

- In $\mathbb{Q}_{\geq 0}$, any element divides any larger element.
- In $\mathbb{N}_0$, any element divides any larger element.
- In $\mathbb{N}_0 \setminus \{1\}$, b divides a if $b + 2 \leq a$ or $b = a$.

A nonzero $a \in M$ is an atom if for any $b, c \in M$ satisfying $a = b + c$, either b or c is 0.

Examples

- There are no atoms of $\mathbb{Q}_{\geq 0}$.
- The only atom of the nonnegative integers is 1.
- In $\mathbb{N}_0 \setminus \{1\}$, the set of atoms is $\{2, 3\}$.

Properties relating to atoms

Definition (Atomic Element)

An element b in a monoid M is atomic if b can be expressed as a sum of atoms or is equal to 0.

Definition (Atomic Monoid)

A monoid is atomic if every element is atomic.

Examples

- Since $\mathbb{Q}_{\geq 0}$ has no atoms, it is not atomic.
- $\mathbb{N}_0$ is atomic because every positive integer is the sum of some amount of 1s.
- $\mathbb{N}_0 \setminus \{1\}$ is also atomic.

Properties relating to atoms, cont.

Definition (Furstenberg)

A monoid M is Furstenberg if for every nonzero element $b \in M$ there exists an atom $a \in M$ such that a divides b.

Definition (Nearly Furstenberg)

A monoid M is nearly Furstenberg if there is an element $c \in M$ such that for every nonzero element $b \in M$ there exists an atom $a \in M$ such that a divides $b + c$ but a does not divide c.

Example

The monoid

$$M := \left\langle \frac{1}{p} \;\middle|\; p \in \mathbb{P}_{\geq 3} \right\rangle \bigcup \mathbb{Q}_{\geq 1}$$

is Furstenberg and is therefore also nearly Furstenberg.

Properties relating to atoms, cont.

Definition (Almost/Quasi-Furstenberg)

A monoid M is almost Furstenberg if for each $b \in M$ there exists an atom $a \in M$ and an atomic element $c \in M$ such that a divides $b + c$ but a does not divide c. Relaxing the condition that each c must be atomic makes the monoid quasi-Furstenberg.

Example

The monoid $M = \left\langle \frac{1}{2}, \frac{1}{3^n} \mid n \in \mathbb{N}_0 \right\rangle$ is quasi-Furstenberg but not almost Furstenberg.

We may abbreviate these as *FM*, *NFM*, *AFM*, and *QFM*.

Motivation

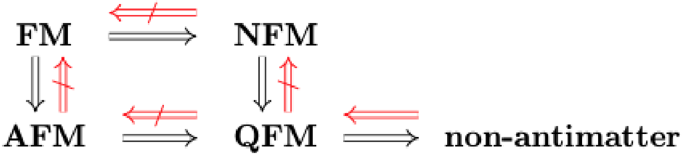
The study of the Furstenberg property is motivated because:

- The Furstenberg property was coined and first studied by P. Clark.
- The Furstenberg property is fundamentally encoded in the celebrated topological proof of the infinitude of primes presented by H. Furstenberg back in 1955.

Our results

Definition (Antimatter)

A monoid is antimatter if it has no atoms.



Both AFM and NFM do not imply Furstenberg

Theorem (Lin-Rabinovitz-Zhang, 2023)

There exists a monoid which is both AFM and NFM but not Furstenberg.

Idea of proof:
The set $\mathbb{N}_0\left[\frac{1}{2}\right]_{>1}$, consisting of all dyadic rationals greater than 1, is countable. Hence we can pick a sequence $(b_n)_{n\geq 1}$ of positive integers and a sequence $(k_n)_{n\geq 1}$ of nonnegative integers such that $\mathbb{N}_0\left[\frac{1}{2}\right]_{>1} = \left\{\frac{b_n}{2^{k_n}} \mid n \in \mathbb{N}\right\}$.
For each $n \in \mathbb{N}$, set $r_n = \frac{b_n}{2^{k_n}}$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of distinct odd primes such that $p_n > b_n$ for every $n \in \mathbb{N}$.
The monoid

$$M := \left\langle \mathbb{N}_0\left[\frac{1}{2}\right] \bigcup \left\{\frac{r_n}{p_n} \;\middle|\; n \in \mathbb{N}\right\}\right\rangle$$

is almost Furstenberg and nearly Furstenberg but not Furstenberg.

QFM does not imply either NFM or AFM

Proposition (Well-known)

A monoid is quasi-Furstenberg if and only if it has at least one atom.

Example

The monoid $M = \left\langle \frac{1}{2}, \frac{1}{3^n} \mid n \in \mathbb{N}_0 \right\rangle$ is quasi-Furstenberg but neither nearly Furstenberg nor almost Furstenberg.

NFM does not imply AFM

Theorem (Lin-Rabinovitz-Zhang, 2023)

The NFM property does not imply the AFM property.

Idea of proof:
For each $p \in \mathbb{P}$ with $p \geq 7$, the monoid

$$M_p := \left\langle \left\{\frac{1}{p}\right\} \bigcup \mathbb{N}_0\left[\frac{1}{2}\right]^{\bullet} \bigcup \left(\frac{1}{2} - \frac{1}{p} + \mathbb{N}_0\left[\frac{1}{2}\right]^{\bullet}\right) \right\rangle$$

is nearly Furstenberg but not almost Furstenberg.

AFM does not imply NFM

Theorem (Lin-Rabinovitz-Zhang, 2023)

The AFM property does not imply the NFM property.

Idea of proof:
Let $(p_n)_{n\geq 1}$ be a sequence whose terms are pairwise distinct odd primes such that $p_n \nmid 2^n - 1$ for every $n \in \mathbb{N}$.
The monoid

$$M := \left\langle \frac{1}{2^n}, \frac{1}{p_n}\left(1 - \frac{1}{2^n}\right) \;\middle|\; n \in \mathbb{N} \right\rangle$$

is almost Furstenberg but not nearly Furstenberg.

Nearly atomic and almost atomic

The definitions for when a monoid is nearly atomic and almost atomic are analogous to the definitions for when a monoid is nearly Furstenberg and almost Furstenberg.

Definition (Nearly Atomic)

A monoid M is nearly atomic if there exists $c \in M$ such that for each nonzero $b \in M$, $b + c$ is atomic.

Definition (Almost Atomic)

A monoid M is almost atomic if for every element $b \in M$ there exists an atomic $c \in M$ such that $b + c$ is atomic.

Proposition (Well-known)

Any nearly atomic monoid is also an almost atomic monoid.

Furstenberg does not imply nearly atomic

Example

The monoid

$$M := \left\langle \frac{1}{p} \;\middle|\; p \in \mathbb{P}_{\geq 3} \right\rangle \bigcup \mathbb{Q}_{\geq 1}$$

is Furstenberg but not almost atomic, which implies that it is not nearly atomic.

Nearly atomic does not imply Furstenberg

Theorem (Lin-Rabinovitz-Zhang, 2023)

There exists a monoid that is nearly atomic but not Furstenberg.

Idea of proof:
For each $x \in \mathbb{N}$, we let $\ell_2(x)$ denote the largest power of 2 less than x. Let $(o_n)_{n \geq 1}$ denote the strictly increasing sequence whose terms are the odd positive integers greater than 1, and let $(p_n)_{n \geq 1}$ denote the strictly increasing sequence whose terms are the primes greater than 3. Notice that $o_i < p_i$ for every $i \in \mathbb{N}$ as prime numbers greater than 3 are a subset of the odd numbers. The monoid

$$M := \left\langle \frac{1}{3}, \frac{1}{2^n}, \frac{o_n}{\ell_2(o_n) p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is nearly atomic but not Furstenberg.

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Thanks!

Thank you for listening.

References

- P. Clark, The Euclidean Criterion for Irreducibles, Amer. Math. Monthly 124 (2017) 198–216.
- P. M. Cohn: *Bezout rings and and their subrings*, Proc. Cambridge Philos. Soc. **64** (1968) 251–264.
- H. Furstenberg, On the infinitude of primes, Amer. Math. Monthly 62 (1955) 353.
- F. Gotti and M. Zafrullah, Integral domains and the IDF property, J. Algebra 614 (2023) 564–591.
- A. Grams: Atomic domains and the ascending chain condition for principal ideals, Math. Proc. Cambridge Philos. Soc. 75 (1974) 321–329.
- N. Lebowitz-Lockard: *On domains with properties weaker than atomicity*, Comm. Algebra **47** (2019) 1862–1868.
- A. Lin, H. Rabinovitz, and Q. Zhang: *The Furstenberg property in Puiseux monoids*, arXiv:2309.12372 [math.AC]