An Extension of Benson's Conjecture to Finite 3-Groups for Monomial Modules with Null Inner Partition

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Introduction to Representation Theory

Representation Theory focuses on using **linear algebra** tools to study **abstract algebraic** objects.

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Definition (Representation)

A **representation** (or **module**) of a finite group G is a vector space V (over a base field k) with a group action ρ, a map from G to the set of bijective linear transformations from V to itself. In particular, for all $g_1, g_2 \in G$,

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2).$$

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For $g \in G$ and $v \in V$, we denote $\rho(g)(v)$ with gv.

Introduction to Representation Theory

Example (180° rotation)

For $G = \mathbb{Z}_2 = \{e, a\}$ with $a^2 = e$, a representation of G over $V = \mathbb{R}^2$ could have ρ with $\rho(e)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\rho(a)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$.

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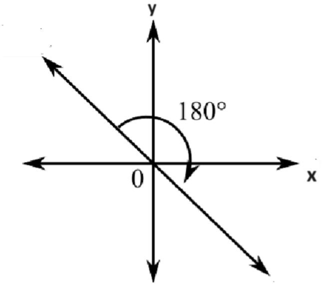
$\rho(a)$ geometric interpretation:



Example (Roots of unity)

For $G = \mathbb{Z}_n = \{e, a, \ldots, a^{n-1}\}$ with $a^n = e$, a representation over $\mathbb{C}$ has group action ρ given by

$$\rho(a^k) = e^{\frac{2\pi ik}{n}},$$

for $k = 0, 1, \ldots, n-1$.

Definition (Subrepresentation)

Let W denote a subspace of a representation V. Then we say W is a **subrepresentation** of V if and only if it is closed under all actions of V.

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Example

For the "180° rotation" representation described previously, a subrepresentation would be the subspace of all vectors $\begin{bmatrix} x \\ 0 \end{bmatrix}$.

Introduction to Representation Theory

Definition (Direct Sum)

For two representations V_α and V_β over group G, the **direct sum** of V_α and V_β has vector space $V_\alpha \oplus V_\beta$ (direct sum as vector spaces) and group action defined by

$$g(v_\alpha \oplus v_\beta) = g(v_\alpha) \oplus g(v_\beta).$$

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Example

Let $V = k$ be a 1-dimensional representation over the base field. Then, for all $v \in V = k$, $V \oplus V$ has group action

$$\rho : v \to \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}.$$

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Definition (Indecomposable/Irreducible)

For a representation V of group G, V is said to be **indecomposable** if it cannot be expressed as a direct sum of two nonzero subrepresentations.

Tensor Product

Definition (Tensor Product)

The **tensor product** $V \otimes W$ is a "multiplication" operation for two vector spaces V and W over a common field k. The following properties hold for all $v \in V$ and $w \in W$ and scalar $a \in k$:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w.$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2.$$
$$av \otimes w = a(v \otimes w).$$
$$v \otimes aw = a(v \otimes w).$$

Monomial Representations

Let k denote a closed field of characteristic 3 and define $G := \mathbb{Z}/3^r\mathbb{Z} \times \mathbb{Z}/3^s\mathbb{Z}$, (a finite 3-group) for integers r and s, with two generators, called x and y.

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Let $[a_1, a_2, \ldots, a_n]/[b_1, b_2, \ldots, b_n]$ denote the partition $[a_1, a_2, \ldots, a_n]$ with the sub-partition $[b_1, b_2, \ldots, b_n]$ "carved out."

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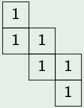
Example

The monomial diagram below corresponds to $[4, 3, 2]/[2, 1, 0]$.



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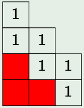

Monomial Representations

In a monomial representation V:

- Each cell is a one-dimensional vector space generated by a basis element of the representation V.
- For a cell in position (a, b), we denote its basis element by $v_{a-1,b-1}$.
- Actions of x and y take basis element $v_{a-1,b-1}$ to cells immediately to the right and above, respectively.

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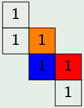
Example

For $[4, 3, 2]/[2, 1, 0]$, $x \cdot (v_{1,1}) = v_{2,1}$ and $y \cdot (v_{1,1}) = v_{1,2}$.



Indecomposable Monomial Representations

Definition (Connected)

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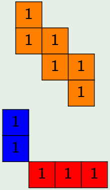

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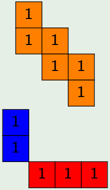

Theorem (Well-known)

The monomial diagram of a monomial representation is connected if and only if it is indecomposable.

Motivation for Research Problem

For finite 2-groups (k a field with characteristic 2 and $G := \mathbb{Z}/2^r\mathbb{Z} \times \mathbb{Z}/2^s\mathbb{Z}$), Benson conjectured the following:

Conjecture (Benson, 2020)

The monomial representation V corresponding to $[a_1, a_2, \ldots, a_n]/[b_1, b_2, \ldots, b_n]$ has a unique odd dimensional indecomposable summand in all its tensor powers if and only if the dimension of V is odd.

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We ask an analogous question for finite 3 groups:

Question

- *For what monomial representations V does $V^{\otimes n}$ have a unique indecomposable summand with dimension nondivisible by 3?*

Uniqueness of Summand

Definition (Dual Representation)

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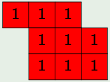
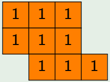
Example

The red and orange monomial diagrams below are duals of each other.





Uniqueness of Summand

Theorem (Well-known)

$V^{\otimes n}$, for all positive integers n, has a unique indecomposable summand with dimension nondivisible by p if $V \otimes V^*$ can be decomposed into a direct sum of k and other indecomposable subrepresentations whose dimensions are divisible by p.

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Using MAGMA, we can use this theorem to our advantage! We use this condition to test whether V has this unique summand as the above are all operations that can be performed in MAGMA (tensor product, decomposition, etc).

Uniqueness of Summand

Recall:

Conjecture (Benson, 2020)

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Conjecture (Proposed Extension of Benson's Conjecture to Finite 3-Groups)

The monomial representation V corresponding to $[a_1, a_2, \ldots, a_n]/[b_1, b_2, \ldots, b_n]$ has a unique indecomposable summand with dimension nondivisible by 3 in all its tensor powers if and only if the dimension of V is nondivisible by 3.

Uniqueness of Summand

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Question

Can we characterize all counterexamples to this extension of Benson's Conjecture? For what monomial representations $[a_1, a_2, \ldots, a_n]/[b_1, b_2, \ldots, b_n]$ *with dimension nondivisible by* 3 *does there not exist a unique indecomposable summand with dimension nondivisible by* 3 *in all its tensor powers?*

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We focus on the case where $b_1 = b_2 = \cdots = b_n = 0$ (a null inner partition).

Uniqueness of Summand

From computational evidence, we propose the following:

Conjecture (Characterization of Counterexamples to Benson's Extension to Finite 3-Groups)

In the case of null inner partition, the monomial representation corresponding to $[a_1, a_2, \ldots, a_n]$ with dimension nondivisible by 3 (equivalently, $\sum_{i=1}^{n} a_i \equiv 1, 2 \pmod 3$) is a counterexample to the proposed extension of Benson's Conjecture if and only if one of the following is true:

- *For $1 \leq i \leq n$, $a_i \equiv 0, 5 \pmod 9$.*
- *For $1 \leq i \leq n$, $a_i \equiv 0, 4 \pmod 9$.*

Uniqueness of Summand

Recall:

Theorem (Well-known)

$V^{\otimes n}$, for all positive integers n, has a unique indecomposable summand with dimension nondivisible by p if $V \otimes V^*$ can be decomposed into a direct sum of k and other indecomposable subrepresentations whose dimensions are divisible by p.

Uniqueness of Summand

In fact, we propose the following even stronger result, which shows one side of the conjecture.

Theorem (Stronger)

Let V_4 denote the monomial representation corresponding to [4], and let V denote a monomial representation corresopnding to an inner-null partition $[a_1, a_2, \ldots, a_n]$ satisfying either

- *for $1 \leq i \leq n$, $a_i \equiv 0, 5 \pmod 9$.*
- *for $1 \leq i \leq n$, $a_i \equiv 0, 4 \pmod 9$.*

Then $V_4 \otimes V_4^* \cong k \oplus M_3 \oplus M_5 \oplus M_7$, where M_3, M_5, M_7 denote subrepresentations of dimension $3, 5, 7$ corresponding to the monomial diagrams shown on the next slide. Furthermore, $V_4 \otimes V_4^*$ is in the decomposition of $V \otimes V^*$ (and thus specifically M_5 and M_7 are in the decomposition as well).

Uniqueness of Summand

k	M_3	M_5	M_7
1	1 1 1	1 1 1 1 1	1 1 1 1 1 1 1

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I would like to thank:

- Dr. Kent Vashaw for his wonderful mentorship and guidance throughout this project.
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- Prof. Benson for giving advice on further directions to pursue.
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