Simple Racks over the Alternating Groups

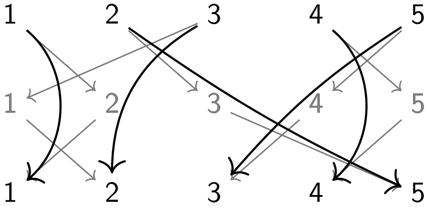
Joseph Vulakh

Under the mentorship of
Prof. Julia Plavnik and Dr. Héctor Peña Pollastri

October 14, 2023
MIT PRIMES Conference

Permutations

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$
- $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}$



- $\sigma\pi = \sigma \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}$
- $\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{pmatrix} \neq \sigma\pi$
- $\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$

The Symmetric Group

$\mathbb{S}_n$ denotes the set of permutations of $\{1, \ldots, n\}$.

Properties of Permutations

For all π, σ, τ in $\mathbb{S}_n$:

- $\pi \circ (\sigma \circ \tau) = (\pi \circ \sigma) \circ \tau$
- $\pi \circ \mathrm{id} = \mathrm{id} \circ \pi = \pi$
- There exists π^{-1} with $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \mathrm{id}$

Any set with a operation satisfying these properties is called a group.

Parity of Permutations

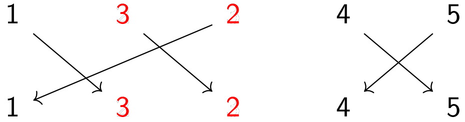
- Every permutation can be written as a product of transpositions
- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = \tau_{12}\tau_{23}\tau_{45}$

  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix} = \tau_{12}\tau_{25}$
- The parity of the permutation is the parity of the number of transpositions
- $\pi = \tau_{12}\tau_{23}\tau_{45} = \tau_{13}\tau_{12}\tau_{45} = \tau_{13}\tau_{12}\tau_{34}\tau_{34}\tau_{45}$
- π is odd, σ is even
- The product of even permutations is even

The Alternating Group

- $\mathbb{A}_n$ denotes the set of even permutations of $\{1, \ldots, n\}$
- $\mathbb{A}_n$ is a group!
- For $n \geq 5$, the group $\mathbb{A}_n$ is simple
  - A simple group is a group with no nontrivial quotients
  - No nontrivial subgroup of $\mathbb{A}_n$ is preserved by conjugation

Conjugation

- 

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$
- $\pi'(\sigma(x)) = \sigma(\pi(x))$
- $\pi' = \sigma\pi\sigma^{-1}$
- Elements of this form are conjugates of π

Conjugation Rack

Let $\sigma \triangleright \pi = \sigma\pi\sigma^{-1}$. Then
$$\begin{aligned}\tau \triangleright (\sigma \triangleright \pi) &= \tau\sigma\pi\sigma^{-1}\tau^{-1} \\ &= \tau\sigma\tau^{-1}\tau\pi\tau^{-1}\tau\sigma^{-1}\tau^{-1} \\ &= \tau\sigma\tau^{-1}\tau\pi\tau^{-1}\tau\sigma^{-1}\tau^{-1} \\ &= (\tau \triangleright \sigma) \triangleright (\tau \triangleright \pi).\end{aligned}$$

Properties of $\triangleright$

- For all π, σ, τ, $\quad \tau \triangleright (\sigma \triangleright \pi) = (\tau \triangleright \sigma) \triangleright (\tau \triangleright \pi)$
- For all σ, τ, there is a unique π such that $\sigma \triangleright \pi = \tau$

Any set with a operation satisfying these properties is called a rack.

Conjugacy classes of a group form racks.

Racks in Research

- Pointed Hopf algebras are important algebraic structures
- Research aims to classify finite-dimensional pointed Hopf algebras
- Racks are important!
- Pointed Hopf algebras can be constructed from finite racks

Question

Can we easily determine whether a pointed Hopf algebra constructed from a rack is finite-dimensional?

Type D

- If a rack is of type D, pointed Hopf algebras constructed from it are infinite-dimensional
- It makes sense to attempt to classify finite racks of type D
- Simple racks are "building blocks" for racks
  - A simple rack is a rack with no nontrivial quotients
- Simple racks can be constructed from simple groups

Question

Can we determine whether a simple rack constructed from $\mathbb{A}_n$ is of type D?

Previously Unsolved Cases

n	ℓ	Cycle type of ℓ	t
any	id	(1^n)	odd, $\gcd(t, n!) = 1$
5		(1^5)	4
5	involution	$(1, 2^2)$	4, odd
6		$(1^2, 2^2)$	odd
8		(2^4)	odd
any	order 4	$(1^{r_1}, 2^{r_2}, 4^{r_4})$ with $r_4 > 0$, $r_2 + r_4$ even	2

n	Cycle type of $\ell(1\ 2)$	t
any	$(1^{s_1}, 2^{s_2}, \ldots, n^{s_n})$ with $s_1 \leq 1$, $s_2 = 0$, $s_h \geq 1$ for some h with $3 \leq h \leq n$	any
	$(1^{s_1}, 2^{s_2}, 4^{s_4})$ with $s_1 \leq 2$ or $s_2 \geq 1$, $s_2 + s_4$ odd, $s_4 \geq 1$	2
5	$(1^3, 2)$	2, 4
6	$(1^4, 2)$	2
	(2^3)	2
7	$(1, 2^3)$	2, odd
8	$(1^2, 2^3)$	odd
10	(2^5)	odd

Our Results

n	ℓ	Cycle type of ℓ	t
any	id	(1^n)	odd, $\gcd(t, n!) = 1$
5		(1^5)	4
5	involution	$(1, 2^2)$	4, odd
6		$(1^2, 2^2)$	odd
8		(2^4)	odd
any	order 4	$(1^{r_1}, 2^{r_2}, 4^{r_4})$ with $r_4 > 0$, $r_2 + r_4$ even	2

n	Cycle type of $\ell(1\ 2)$	t
any	$(1^{s_1}, 2^{s_2}, \ldots, n^{s_n})$ with $s_1 \leq 1$, $s_2 = 0$, $s_h \geq 1$ for some h with $3 \leq h \leq n$	any
	$(1^{s_1}, 2^{s_2}, 4^{s_4})$ with $s_1 \leq 2$ or $s_2 \geq 1$, $s_2 + s_4$ odd, $s_4 \geq 1$	2
5	$(1^3, 2)$	2, 4
6	$(1^4, 2)$	2
	(2^3)	2
7	$(1, 2^3)$	2, odd
8	$(1^2, 2^3)$	odd
10	(2^5)	odd

Acknowledgements

- Prof. Julia Plavnik and Dr. Héctor Peña Pollastri
- MIT PRIMES
- My family

References

[1] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. "Pointed Hopf Algebras". In: *New Directions in Hopf Algebras*. Cambridge University Press. 2002, pp. 1–68.

[2] Nicolás Andruskiewitsch et al. "Finite-dimensional pointed Hopf algebras with alternating groups are trivial". In: *Annali di Matematica Pura ed Applicata* 190.2 (2011), pp. 225–245.

[3] Nicolás Andruskiewitsch et al. "On twisted homogeneous racks of type D". In: *Revista de la Union Matematica Argentina* 51.2 (2010), pp. 1–16.

[4] Joseph Vulakh. "Twisted homogeneous racks over the alternating groups". In: *AMS Contemporary Mathematics* (to appear). URL: https://arxiv.org/abs/2305.19136.