On the Classification of Low-Rank Odd-Dimensional Modular Categories

William Gvozdjak
Mentors: Prof. Julia Plavnik and Agustina Czenky

MIT PRIMES Conference

October 14, 2023

Research Motivation and Question

- **Modular categories** (modular tensor categories, MTCs): concepts of interest in various fields
  - Quantum computing, topological quantum field theories, quantum groups, etc.

Research Motivation and Question

- **Modular categories** (modular tensor categories, MTCs): concepts of interest in various fields
  - Quantum computing, topological quantum field theories, quantum groups, etc.
- What MTCs exist? (classify them)

Research Motivation and Question

- **Modular categories** (modular tensor categories, MTCs): concepts of interest in various fields
  - Quantum computing, topological quantum field theories, quantum groups, etc.
- What MTCs exist? (classify them)
- Look at a specific kind of modular categories: **odd-dimensional** ones

Research Motivation and Question

- **Modular categories** (modular tensor categories, MTCs): concepts of interest in various fields
  - Quantum computing, topological quantum field theories, quantum groups, etc.
- What MTCs exist? (classify them)
- Look at a specific kind of modular categories: **odd-dimensional** ones
- Classify them by **rank** (similar idea to dimension of a vector space)

Research Motivation and Question

- **Modular categories** (modular tensor categories, MTCs): concepts of interest in various fields
  - Quantum computing, topological quantum field theories, quantum groups, etc.
- What MTCs exist? (classify them)
- Look at a specific kind of modular categories: **odd-dimensional** ones
- Classify them by **rank** (similar idea to dimension of a vector space)

Research Goal

Advance the classification of odd-dimensional MTCs by rank.

Introductory Example of a Category: **k**-Vec

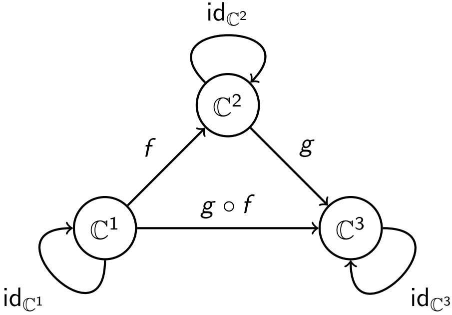
- Consider all vector spaces over a field **k**.
  - Work over an algebraically closed field **k** of characteristic 0 (e.g., $\mathbb{C}$, not $\mathbb{R}$).

Introductory Example of a Category: **k**-Vec

- Consider all vector spaces over a field **k**.
  - Work over an algebraically closed field **k** of characteristic 0 (e.g., $\mathbb{C}$, not $\mathbb{R}$).
- The category **k**-Vec has:
  - Vector spaces themselves. Call them "objects."
  - Linear maps between vector spaces. Call them "morphisms."

Introductory Example of a Category: $\mathbf{k}$-Vec

- Consider all vector spaces over a field $\mathbf{k}$.
  - Work over an algebraically closed field $\mathbf{k}$ of characteristic 0 (e.g., $\mathbb{C}$, not $\mathbb{R}$).
- The category $\mathbf{k}$-Vec has:
  - Vector spaces themselves. Call them "objects."
  - Linear maps between vector spaces. Call them "morphisms."
- The following is an incomplete drawing of the category $\mathbb{C}$-Vec.



Definition of a Category

Definition

A **category** contains:

- a collection of objects $X, Y, Z, \dots,$
- a collection of morphisms $f, g, h, \dots$ between those objects,

such that each object has an identity morphism, morphisms compose, and morphisms are associative.

Example

In the category **k**-Vec, we have:

- Objects as vector spaces over **k**, and
- Morphisms as linear maps between vector spaces.

Tensor Categories

A tensor category adds two operations to a category, allowing us to "add" and "multiply" objects.

Tensor Categories

A tensor category adds two operations to a category, allowing us to "add" and "multiply" objects.

Definition (loose)

Loosely, a **tensor category** $\mathcal{C}$ is a category with the following:

- Direct sum $\oplus$ of two objects: $X \oplus Y$.
- Tensor product $\otimes$ of two objects: $X \otimes Y$. Comes with:
  - Unit object $\mathbb{1}$: $\mathbb{1} \otimes X \xrightarrow{\sim} X$ and $X \otimes \mathbb{1} \xrightarrow{\sim} X$
  - Associativity constraints $\alpha_{X,Y,Z}$ for all objects X, Y, Z: $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$.
- Additional conditions (omitted)

Fusion Categories

Intuitively, a fusion category is a tensor category with a "finite basis."

Fusion Categories

Intuitively, a fusion category is a tensor category with a "finite basis."

Definition

A **simple object** in a tensor category is one whose only subobjects with respect to the direct sum are 0 and itself.

Fusion Categories

Intuitively, a fusion category is a tensor category with a "finite basis."

Definition

A **simple object** in a tensor category is one whose only subobjects with respect to the direct sum are 0 and itself.

Definition

A tensor category is **semisimple** if all its objects can be expressed as a direct sum of finitely many simple objects X_i:

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_k.$$

Fusion Categories

Intuitively, a fusion category is a tensor category with a "finite basis."

Definition

A **simple object** in a tensor category is one whose only subobjects with respect to the direct sum are 0 and itself.

Definition

A tensor category is **semisimple** if all its objects can be expressed as a direct sum of finitely many simple objects X_i:

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_k.$$

Definition

A **fusion category** is a semisimple tensor category with finitely many simple objects. Its **rank** is the number of simple objects it has.

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g \in G} V_g$.

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g \in G} V_g$.

- Direct sum: $V \oplus W = \bigoplus_{g \in G}(V_g \oplus W_g)$

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g \in G} V_g$.

- Direct sum: $V \oplus W = \bigoplus_{g \in G} (V_g \oplus W_g)$
  - Simple objects: for each $h \in G$, δ_h such that $(\delta_h)_g = \begin{cases} \mathbf{k} & h = g \\ 0 & h \neq g \end{cases}$

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g\in G} V_g$.

- Direct sum: $V \oplus W = \bigoplus_{g\in G}(V_g \oplus W_g)$
  - Simple objects: for each $h \in G$, δ_h such that $(\delta_h)_g = \begin{cases} \mathbf{k} & h = g \\ 0 & h \neq g \end{cases}$
- Tensor product: $(V \otimes W)_g = \bigoplus_{\substack{x,y\in G \\ xy=g}}(V_x \otimes W_y)$ for all $g \in G$
  - Unit object: δ_e
  - Associativity constraint: given by a 3-cocycle $\omega : G \times G \times G \to \mathbf{k}^\times$

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g \in G} V_g$.

- Direct sum: $V \oplus W = \bigoplus_{g \in G}(V_g \oplus W_g)$
  - Simple objects: for each $h \in G$, δ_h such that $(\delta_h)_g = \begin{cases} \mathbf{k} & h = g \\ 0 & h \neq g \end{cases}$
- Tensor product: $(V \otimes W)_g = \bigoplus_{\substack{x,y \in G \\ xy = g}}(V_x \otimes W_y)$ for all $g \in G$
  - Unit object: δ_e
  - Associativity constraint: given by a 3-cocycle $\omega : G \times G \times G \to \mathbf{k}^\times$
- Semisimple: all objects are finite-dimensional vector spaces

$\mathbf{k}\text{-Vec}_G^\omega$

Example

The category $\mathbf{k}\text{-Vec}_G^\omega$ of finite-dimensional G-graded vector spaces over a field $\mathbf{k}$, where G is a group of finite order and ω is a 3-cocycle, is a fusion category.

Objects in $\mathbf{k}\text{-Vec}_G^\omega$ are also vector spaces, but they are now **graded** by the group G: $V = \bigoplus_{g \in G} V_g$.

- Direct sum: $V \oplus W = \bigoplus_{g \in G} (V_g \oplus W_g)$
  - Simple objects: for each $h \in G$, δ_h such that $(\delta_h)_g = \begin{cases} \mathbf{k} & h = g \\ 0 & h \neq g \end{cases}$
- Tensor product: $(V \otimes W)_g = \bigoplus_{\substack{x,y \in G \\ xy = g}} (V_x \otimes W_y)$ for all $g \in G$
  - Unit object: δ_e
  - Associativity constraint: given by a 3-cocycle $\omega : G \times G \times G \to \mathbf{k}^\times$
- Semisimple: all objects are finite-dimensional vector spaces
- Finitely many simples: G has finite order

Modular Categories

Definition

A **modular category** is a fusion category equipped with spherical and braiding structures that satisfies a non-degeneracy condition. Modular categories are also called modular tensor categories (MTCs).

Vaguely, the braiding makes the tensor product "commute" in a coherent way.

Modular Categories

Definition

A **modular category** is a fusion category equipped with spherical and braiding structures that satisfies a non-degeneracy condition. Modular categories are also called modular tensor categories (MTCs).

Vaguely, the braiding makes the tensor product "commute" in a coherent way.

Why do we care?

Example (why do we care?)

Topological quantum computing is an approach to quantum computing using **anyons**. Anyon systems are modelled by unitary modular categories, where anyons are represented by the category's simple objects.

Frobenius-Perron Dimension

An important property of an object X in a fusion category is its **Frobenius-Perron dimension**, denoted $\mathrm{FPdim}(X)$ (definition omitted). It is a nonnegative real number.

For any objects X and Y, it satisfies the following equations:

- $\mathrm{FPdim}(\mathbb{1}) = 1$,
- $\mathrm{FPdim}(X \oplus Y) = \mathrm{FPdim}(X) + \mathrm{FPdim}(Y)$,
- $\mathrm{FPdim}(X \otimes Y) = \mathrm{FPdim}(X) \cdot \mathrm{FPdim}(Y)$.

Frobenius-Perron Dimension

An important property of an object X in a fusion category is its **Frobenius-Perron dimension**, denoted $\operatorname{FPdim}(X)$ (definition omitted). It is a nonnegative real number.

For any objects X and Y, it satisfies the following equations:

- $\operatorname{FPdim}(\mathbb{1}) = 1$,
- $\operatorname{FPdim}(X \oplus Y) = \operatorname{FPdim}(X) + \operatorname{FPdim}(Y)$,
- $\operatorname{FPdim}(X \otimes Y) = \operatorname{FPdim}(X) \cdot \operatorname{FPdim}(Y)$.

Definition

The **Frobenius-Perron dimension** of a fusion category $\mathcal{C}$ is defined by

$$\operatorname{FPdim}(\mathcal{C}) = \sum_{X \text{ simple}} \operatorname{FPdim}(X)^2.$$

We say a fusion category with odd Frobenius-Perron dimension is **odd-dimensional**.

Pointed and Perfect Fusion Categories

Definition

An object in a fusion category is **invertible** if its Frobenius-Perron dimension is 1.

Pointed and Perfect Fusion Categories

Definition

An object in a fusion category is **invertible** if its Frobenius-Perron dimension is 1.

Definition

A fusion category is **pointed** if all its simple objects are invertible.

Pointed and Perfect Fusion Categories

Definition

An object in a fusion category is **invertible** if its Frobenius-Perron dimension is 1.

Definition

A fusion category is **pointed** if all its simple objects are invertible.

Definition

A fusion category is **perfect** if its only invertible simple object is the unit $\mathbb{1}$.

Previous Results

Research Goal

Advance the classification of odd-dimensional modular categories by rank.

Previous Results

Research Goal

Advance the classification of odd-dimensional modular categories by rank.

Theorem (see, for example, Etingof, Nikshych, and Ostrik, 2005)

Pointed fusion categories are classified by finite groups G and 3-cocycles ω: if $\mathcal{C}$ is pointed, then $\mathcal{C} \cong \mathbf{k}\text{-Vec}_G^{\omega}$.

This was shown for braided tensor categories by Joel and Street, 1993.

Previous Results

Research Goal

Advance the classification of odd-dimensional modular categories by rank.

Theorem (see, for example, Etingof, Nikshych, and Ostrik, 2005)

Pointed fusion categories are classified by finite groups G and 3-cocycles ω: if $\mathcal{C}$ is pointed, then $\mathcal{C} \cong \mathbf{k}\text{-}\mathrm{Vec}_G^\omega$.

This was shown for braided tensor categories by Joel and Street, 1993.

Theorem (Czenky and Plavnik, 2022)

All odd-dimensional MTCs of rank at most 15 are pointed. All odd-dimensional MTCs of rank 17, 19, 21, and 23 are either pointed or perfect.

Previous Results

Research Goal

Advance the classification of odd-dimensional modular categories by rank.

Theorem (see, for example, Etingof, Nikshych, and Ostrik, 2005)

Pointed fusion categories are classified by finite groups G and 3-cocycles ω: if $\mathcal{C}$ is pointed, then $\mathcal{C} \cong \mathbf{k}\text{-Vec}_G^\omega$.

This was shown for braided tensor categories by Joel and Street, 1993.

Theorem (Czenky and Plavnik, 2022)

All odd-dimensional MTCs of rank at most 15 are pointed. All odd-dimensional MTCs of rank 17, 19, 21, and 23 are either pointed or perfect.

Theorem (Bruillard and Rowell, 2012)

There exists a non-pointed odd-dimensional MTC of rank 25, $\text{Rep}(D^\omega(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$.

Our Results

Research Question

Are odd-dimensional MTCs of rank less than 25 necessarily pointed?

Our Results

Research Question

Are odd-dimensional MTCs of rank less than 25 necessarily pointed?

Theorem

All odd-dimensional MTCs of rank 17, 19, 21, *and* 23 *are pointed.*

Thus, all odd-dimensional MTCs of rank less than 25 are pointed. This implies that $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ is the lowest-rank non-pointed odd-dimensional MTC.

Our Results

Research Question

Are odd-dimensional MTCs of rank less than 25 necessarily pointed?

Theorem

All odd-dimensional MTCs of rank 17, 19, 21, *and* 23 *are pointed.*

Thus, all odd-dimensional MTCs of rank less than 25 are pointed. This implies that $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ is the lowest-rank non-pointed odd-dimensional MTC.

Theorem

All odd-dimensional MTCs of rank 25 *are pointed, perfect, or* $\mathrm{Rep}(D^{\omega}(\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$.

We also showed additional results for odd-dimensional MTCs up to rank 73.

Acknowledgements

I would like to thank:

- My mentors Prof. Julia Plavnik and Agustina Czenky for suggesting this topic to me and all their guidance throughout the project
- Prof. Etingof, Dr. Gerovitch, Dr. Khovanova, and the MIT PRIMES-USA program for making this incredible research opportunity possible
- My parents for their encouragement

References

- P. Bruillard, E. Rowell. *Modular categories, integrality and egyptian fractions*. Proceedings of the American Mathematical Society **140** (2012), no. 4, 1141–1150.
- A. Czenky, J. Plavnik. *On odd-dimensional modular tensor categories*. Algebra & Number Theory **16** (2022), no. 8, 1919–1939.
- P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs **205** (2015), AMS.
- P. Etingof, D. Nikshych, V. Ostrik. *On fusion categories*. Annals of Mathematics **162** (2005), 581–642.
- A. Joyal, R. Street. *Braided tensor categories*. Advances in Mathematics **102** (1993), 20–78.
- E. Riehl. *Category Theory in Context*. Dover Publications, 2016.
- E. Rowell. *Braids, Motions and Topological Quantum Computing*. Preprint `arXiv 2208.11762` (2022).