Extremal Bounds on Peripherality Measures

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Conventions

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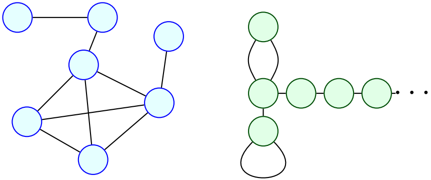

Finite simple graphs, usually connected. The name of the graph is always G and the number of vertices is always denoted n.

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A peripherality measure is the opposite of a centrality measure; peripheral vertices are the least important in a graph.

Applications

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- Atmospheric networks

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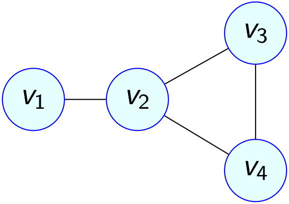

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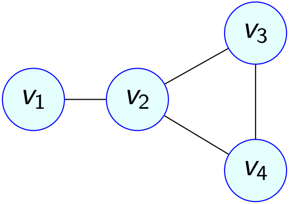

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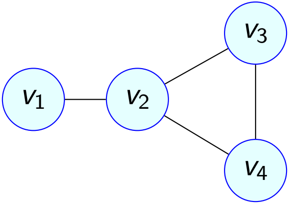

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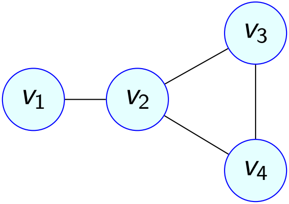

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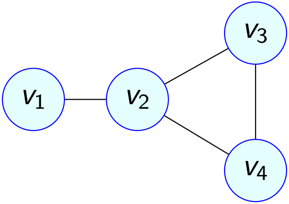

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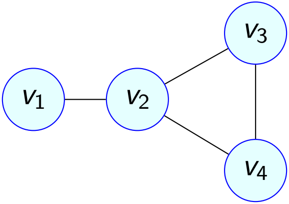

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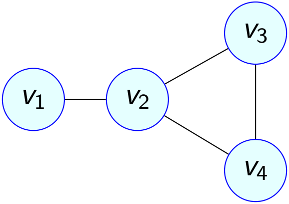

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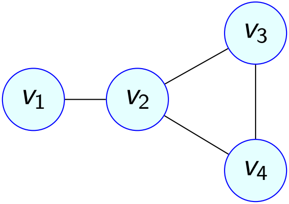

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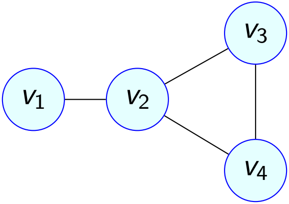

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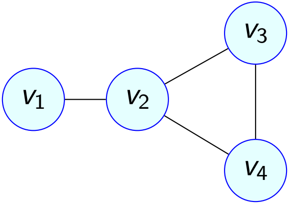

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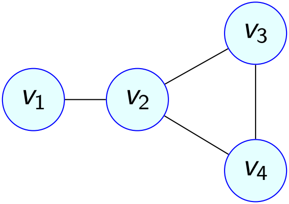
Peripherality Example



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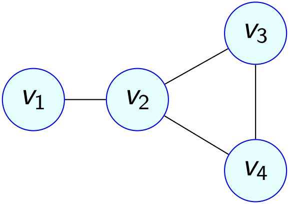

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$$\mathrm{peri}(v_1) = 0$$

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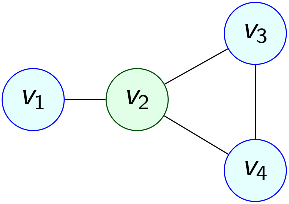

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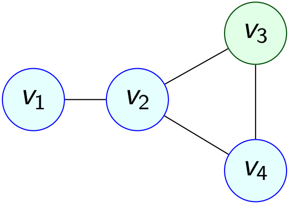

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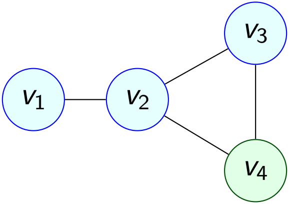

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Theorem

The peripherality of a graph is the number of unordered pairs (v, x) of vertices such that $n_G(v, x) \neq n_G(x, v)$.

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Corollary

The peripherality of an n-vertex graph is at most $\binom{n}{2}$.

Geneson and Tsai found constructions of the equality case for each $n \geq 9$. We determined the maximum for each $n < 9$.

Edge Peripherality

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New bound

The maximum edge peripherality of an n-vertex graph lies in the interval $[\frac{\sqrt{3}}{24}n^3(1 - o(1)), \frac{1}{6}n^3]$.

Edge Sum Peripherality

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The maximum edge sum peripherality of an n-vertex bipartite graph of diameter at most 3 is $\frac{1}{8}n^4 - O(n^2)$.

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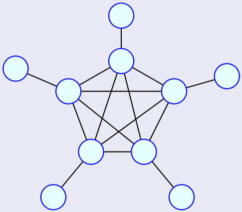
The Trinajstić index of a graph is the sum of $NT(u, v)$ over all $\binom{n}{2}$ pairs.

Trinajstić Index

Conjecture (Furtula)

For sufficiently large n, the Trinajstić index of an n-vertex graph is maximized by the generalization of this graph:



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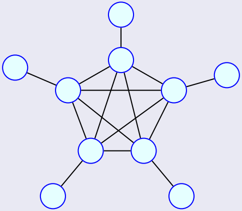

Verdict

The conjecture is false. This family of graphs achieves $NT(G) \leq 0.25n^4(1 + o(1))$. The maximum of $NT(G)$ is actually $0.5n^4(1 - o(1))$.

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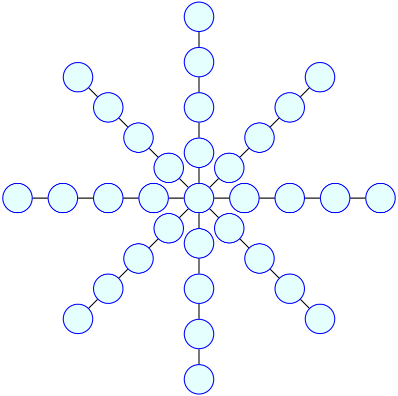

As the number of "arms" and the length of each "arm" both go to infinity, $NT(G) = 0.5n^4(1 - o(1))$.

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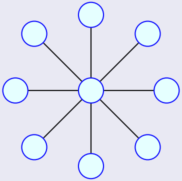
For sufficiently large n, the Trinajstić index of an n-vertex tree is minimized by the generalization of this graph:



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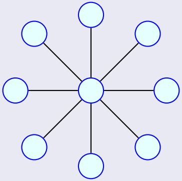

This conjecture is still open.

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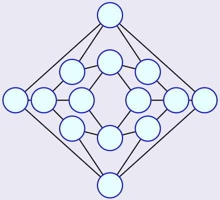

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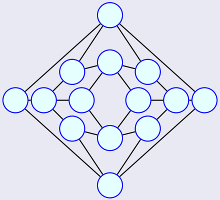

In fact, these can be used to generate arbitrarily large counterexamples.

Acknowledgements

I thank Dr. Jesse Geneson for suggesting this research topic, telling me about many possible directions for research, helping me format my results into a paper, giving me feedback on drafts of the paper, helping me submit it to arXiv and a journal, and giving me feedback on my presentation rehearsal.
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References

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