On Generalized Eulerian Numbers

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Permutations

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- Treat these as functions (bijections) from $\{1, 2, \ldots, n\}$ to itself.

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Example:

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- Here, $\sigma(1) = 5, \sigma(2) = 6, \sigma(3) = 3$, etc.
- Sometimes, we simplify and write 563142.

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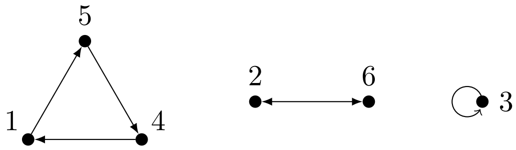

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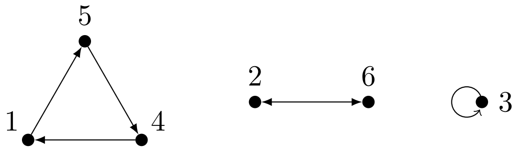

- Each arrow represents an application of σ to the node.
- We similarly use shorthand and write $\sigma = (154)(26)(3)$.
- By convention, we arrange cycles by smallest element, and put smallest element on the left (ensures uniquness!)

Ascents

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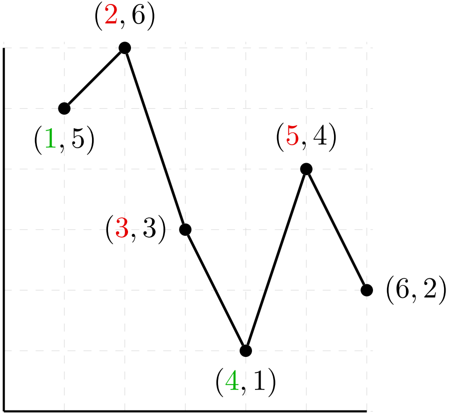

- Ascent indices are marked in green.
- ***Descents*** are whenever $\sigma(i) > \sigma(i+1)$ (indices marked in red).
- Two ascents: ascent of size 1 at $i = 1$, ascent of size 3 at $i = 3$.

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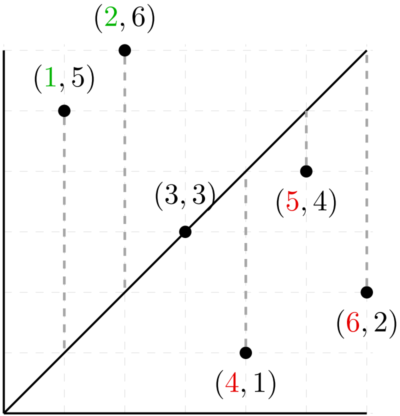

- Excedances are marked in green.
- ***Anti-excedances***, whenever $\sigma(i) < i$, are marked in red.
- Two excedances: an excedance of size 4 at $i = 1$ and $i = 2$.

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The Foata transform:

- Takes a permutation σ in two-line notation.
- Splits the permutation into blocks:
- Stops at every element smaller than all previous elements, and start a new block before that element.
- Creates a new permutation $F(\sigma)$ where every block in σ is interpreted as cycle in $F(\sigma)$.

The Foata Transform

- Example permutation:

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- Stop at every element smaller than all previous elements, and start a new block before that element.
- Interpret blocks as cycles in transformed permutation $F(\sigma)$:

$$F(\sigma) = (56)(3)(142) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}.$$

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- Ascents in σ correspond exactly with excedances in $F(\sigma)$!
- Descents inside blocks also correspond exactly.
- Finally, by convention, there must always be a descent/anti-excedance at the end of blocks.

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Proposition

After an application of the Foata transform on any permutation σ, number of ascents in σ ***always*** *equal to number of excedances in $F(\sigma)$.*

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$$F(\sigma) = (56)(3)(142) \implies \sigma = 563142.$$

- It is therefore a bijection!

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- By the Foata transform, this is **ALSO** the number of permutations with exactly m excedances.
- Example: $E(3, 1) = 4$. Four with exactly one ascent:

  132, 213, 231, 312.

  Four with exactly one excedance:

  132, 213, 312, 321.

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Definition (r-Excedance)

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- Similarly, 1-excedances are equivalent to regular excedances.

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- We claim $E_r(n,m)$ also counts the number of permutations with exactly m r-excedances.
- Consider our old examples:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix}, \qquad F(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}.$$

- Power of Foata transform: ascent size in σ matched exactly with excedance size in $F(\sigma)$.

A Further Generalization

- Inspired by past projects, we defined:

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The number $E_r(n, m, k)$ counts the number of permutations $1, 2, \ldots, n$ with exactly m r-excedances, **and** ends with k (i.e., $\sigma(n) = k$.)

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- Main theorem proven:

Theorem (Dong 2023)

The number $E_r(n, m, k)$ also counts the number of permutations $1, 2, \ldots, n$ with exactly m r-ascents and ends with $n + 1 - k$.

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- We can show that $E_r(n, m, k)$ also counts the permutations with m r-descents and ends with k (somewhat nicer, though in either case symmetry is broken).

A Further Generalization

We also proved several other properties of these numbers, including:

- The following generalization of Worpitzky's identity holds:

$$(x+1)^{n-k+1}x^{k-1} = \sum_{i=0}^{n} E_1(n,i,k)\binom{x+i}{n-1}.$$

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- It is possible to convert this generating function into an explicit formula for $E_1(n,m,k)$.

- For all integers n, m, k with $k \geq 2$, we have the equality:

$$\begin{aligned} E_{r+1}(n,m,k) &= E_r(n,m+1,k-1) + (r-1)E_r(n-1,m,k-1) \\ &\quad - (r-1)E_r(n-1,m+1,k-1). \end{aligned}$$

  Furthermore, $E_{r+1}(n,m,1) = E_r(n,m,n)$.

- This allows us to compute and potentially derive an explicit formula for $E_r(n,m,k)$.

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