Counting the
Involutions of the Symmetric Group

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01 **Background**

02 **Robinson-Schensted Algorithm**

03 **Viennot ‘s Geometric Construction**

number involutions of S_n

$\Leftrightarrow$

std tableaux with n elements

Definitions

Symmetric Group

A collection of the permutations of {1,…, *n*} that acts as a group with composition of permutations being the group operation

There are n! elements

Ex.

We write elements as two rows ⟶ 1 2 3 ⟵omit the first row for simplicity

1 3 2

132 is an element of S_3 where

$1 \longrightarrow 1$

$2 \longrightarrow 3$

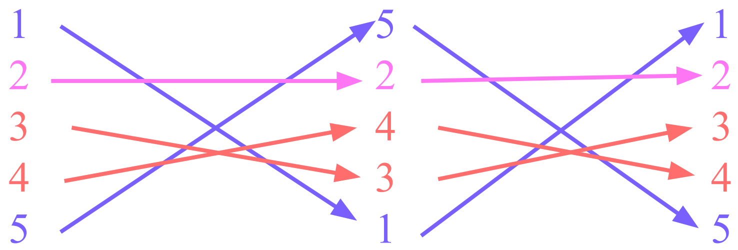
$3 \longrightarrow 2$

Definitions

Involution

For S_n , an element is an involution if applying the permutation twice maps every element to itself

Ex.
52431 in S_5

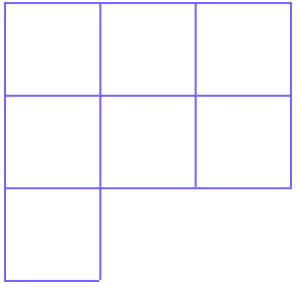


Partition

Unordered decomposition of a positive integer into positive integer parts

eg. 7 = 3 + 3 + 1

Ferrers Diagram



shape λ = (3,3,1)

Tableau

5	3	1
2	4	6
7		

Standard Tableau

Increasing →

Increasing ↓

1	2	3
4	5	6
7		

1	3	5
2	4	6
7		

1	3	4
5	2	7
6		

Standard Tableau

Increasing →

Increasing ↓

1	2	3
4	5	6
7		

standard

1	3	5
2	4	6
7		

standard

1	3	4
5	2	7
6		

not standard

number involutions of S_n

$\Leftrightarrow$

std tableaux with n elements

R-S

Algorithm

$$\pi \Longleftrightarrow (P, Q)$$

permutations in S_n

pairs of std tableaux

Forward Correspondence

P "insertion tableau"

given $\pi = x_1 x_2 \ldots x_n$
for *i* in *1,…, n* {
    R = 1
    $x = x_i$
    while *x* < some element in row R {
        y = min element of row R st *y* > *x*
        replace *y* by *x*
        set *x:= y*
        R++
    }
    append *x* to the end of row R
}

Q "recording tableau"

given $\pi = x_1 x_2 \ldots x_n$
for *i* in *1,…, n* {
    set the cell in which insertion terminates to *n*
}

$\pi = 41325$

INSERTING: 4
DISPLACED: n/a

4

P

Q

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_1
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

inserted: 4
DISPLACED: n/a

P

4

Q

1

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_1
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to 1
}
```

$\pi = 41325$

INSERTING: 1
DISPLACED: 4

P

1

Q

1

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_2
        while x < some element in row 1 {
                y = min element of row 1 st y > x = 4
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

INSERTING: 4
DISPLACED: n/a

P

1
4

Q

1

P TABLEAU

for *i* in *1,…, 5* {
R = 1
$x = x_2$
while *x* < some element in row R {
y = min element of row R st y > x
replace *y* by *x*
set *x:= y*
R++
}
append *x = 4* to the end of row 2
}

Q TABLEAU

for i in 1,…, 5 {
set the cell in which insertion terminates to *n*
}

$\pi = 41325$

inserted: 1
DISPLACED: 4

P

1
4

Q

1
2

P TABLEAU

for *i* in *1,…, 5* {
  R = 1
  $x = x_2$
  while *x* < some element in row R {
    y = min element of row R st y > x
    replace *y* by *x*
    set *x:= y*
    R++
  }
  append *x* to the end of row R
}

Q TABLEAU

for i in 1,…, 5 {
  set the cell in which insertion terminates to *2*
}

$\pi = 41325$

INSERTING: 3
DISPLACED: n/a

P

1	**3**
4	

Q

1
2

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_3
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append 3 to the end of row 1
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

inserted: 3
DISPLACED: n/a

P

1	3
4	

Q

1	3
2	

P TABLEAU

for *i* in *1,…, 5* {
R = 1
$x = x_3$
while *x* < some element in row R {
y = min element of row R st y > x
replace *y* by *x*
set *x:= y*
R++
}
append *x* to the end of row R
}

Q TABLEAU

for i in 1,…, 5 {
set the cell in which insertion terminates to *n*
}

$\pi = 41325$

INSERTING: 2
DISPLACED: 3

P

1	2
4	

Q

1	3
2	

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_4
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

INSERTING: 3
DISPLACED: 4

P

1	2
3	

Q

1	3
2	

P TABLEAU

for *i* in *1,…, 5* {
R = 1
x = x_4
while *x* < some element in row R {
y = min element of row R st y > x
replace *y* by *x*
set *x:= y*
R++
}
append *x* to the end of row R
}

Q TABLEAU

for i in 1,…, 5 {
set the cell in which insertion terminates to *n*
}

$\pi = 41325$

INSERTING: 4
DISPLACED: n/a

P

1	2
3	
4	

Q

1	3
2	

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_4
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

inserted: 2
DISPLACED: 3, 4

P

1	2
3	
4	

Q

1	3
2	
4	

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_4
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

INSERTING: 5
DISPLACED: n/a

P

1	2	5
3		
4		

Q

1	3
2	
4	

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_5
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

inserted: 5
DISPLACED: n/a

P

1	2	5
3		
4		

Q

1	3	**5**
2		
4		

P TABLEAU

```
for i in 1,..., 5 {
        R = 1
        x = x_5
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Done!

Q TABLEAU

```
for i in 1,..., 5 {
        set the cell in which insertion terminates to n
}
```

$\pi = 41325$

INSERTING: //
DISPLACED: //

P

1	2	5
3		
4		

Q

1	3	5
2		
4		

P TABLEAU

```
for i in 1,…, 5 {
        R = 1
        x = x_i
        while x < some element in row R {
                y = min element of row R st y > x
                replace y by x
                set x:= y
                R++
        }
        append x to the end of row R
}
```

Done!

Q TABLEAU

```
for i in 1,…, 5 {
        set the cell in which insertion terminates to n
}
```

Done!

$$\pi \longrightarrow (P, Q)$$

$$(41325) \longrightarrow \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array} \right)$$

$$\pi \overset{?}{\longleftarrow} (P, Q)$$

$$(41325) \overset{?}{\longleftarrow} \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right)$$

$\pi =$ ___ ___ ___ ___ ___

P

1	2	5
3		
4		

Q

1	3	5
2		
4		

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *index*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

    find the largest element *y* of row R-1 less than *x*

    replace *y* by *x*

    set *x* := *y*

    R--

when row one reached, *x* is the index *index* element of the permutation

$\pi =$ ___ ___ ___ ___ ___

P

1	2	5
3		
4		

Q

1	3	5
2		
4		

(1, 3)

From Q:
take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:
take element in *(i, j)* and call it *x*
let row be row R
while row R isn't first row:
    find the largest element *y* of row R-1 less than *x*
    replace *y* by *x*
    set *x := y*
    R--
when row one reached, *x* is the index *idx* element of the permutation

$\pi =$ __ __ __ __ __

P

1	2	5
3		
4		

Q

1	3	5
2		
4		

(1, 3)

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

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while row R isn't first row:

  find the largest element *y* of row R-1 less than *x*

  replace *y* by *x*

  set *x* := *y*

  R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	2	5
3		
4		

Q

1	3	5
2		
4		

(1, 3)

From Q:
take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:
take element in *(i, j)* and call it *x*
let row be row R
while row R isn't first row:
    find the largest element *y* of row R-1 less than *x*
    replace *y* by *x*
    set *x* := *y*
    R--
when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	2
3	
4	

Q

1	3
2	
4	

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

  find the largest element *y* of row R-1 less than *x*

  replace *y* by *x*

  set *x := y*

  R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	2
3	
4	

Q

1	3
2	
4	

(3, 1)

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

  find the largest element *y* of row R-1 less than *x*

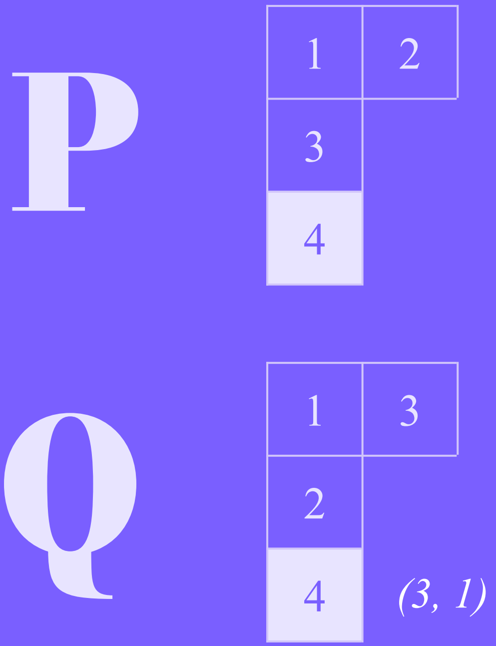
  replace *y* by *x*

  set *x := y*

  R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$



From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

    find the largest element *y* of row R-1 less than *x*

    replace *y* by *x*

    set *x := y*

    R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	2
3	
4	

Q

1	3
2	
4	*(3, 1)*

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

  find the largest element *y* of row R-1 less than *x*

  replace *y* by *x*

  set *x := y*

  R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	2
4	

displaced: 3

Q

1	3
2	
4	

(3, 1)

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

- find the largest element *y* of row R-1 less than *x*
- replace *y* by *x*
- set *x* := *y*
- R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ \_\ 5$

P

1	3
4	

displaced: 2

Q

1	3
2	
4	

(3, 1)

From Q:
take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:
take element in *(i, j)* and call it *x*
let row be row R
while row R isn't first row:

- find the largest element *y* of row R-1 less than *x*
- replace *y* by *x*
- set *x := y*
- R--

when row one reached, *x* is the index *idx* element of the permutation

$\pi = \_\ \_\ \_\ 2\ 5$

P

1	3
4	

displaced: 2

Q

1	3
2	
4	

(3, 1)

From Q:

take *largest cell* and find its coordinates *(i, j)* and content *idx*

In P:

take element in *(i, j)* and call it *x*

let row be row R

while row R isn't first row:

    find the largest element *y* of row R-1 less than *x*

    replace *y* by *x*

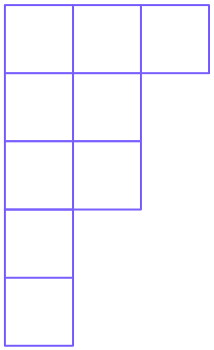
    set *x* := *y*

    R--

when row one reached, *x* is the index *idx* element of the permutation

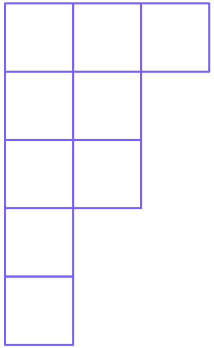
$$\pi \rightleftarrows (P, Q)$$

$$(41325) \rightleftarrows \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right)$$

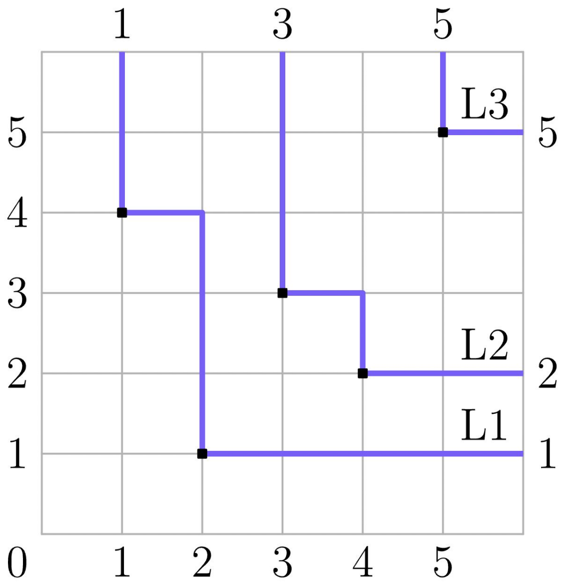


Viennot's Construction

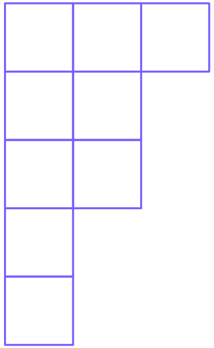
Correspondence to a Shadow Line Diagram

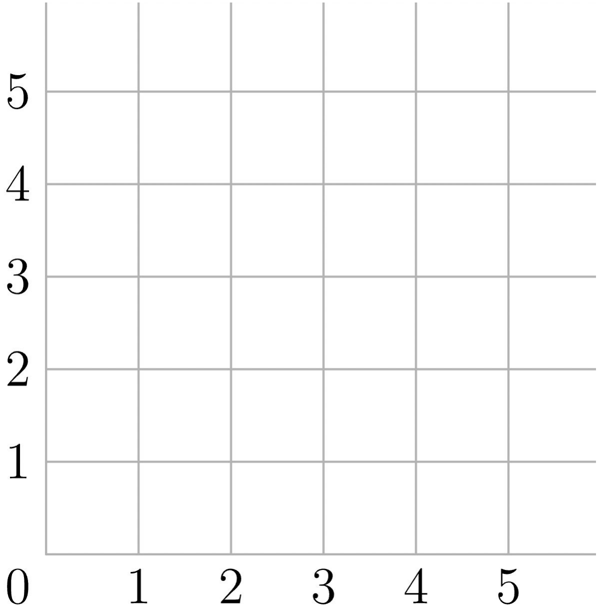


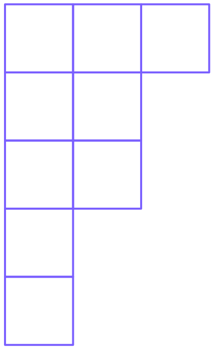
$\pi \Leftrightarrow$



Making a shadow diagram (for $\pi = 41325$):

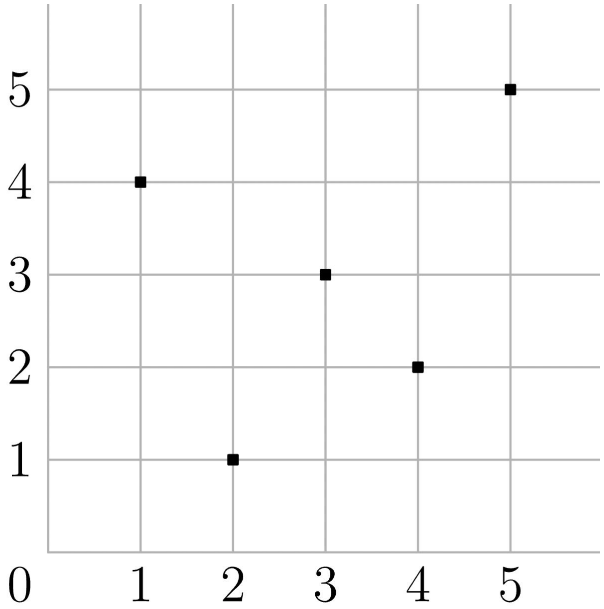


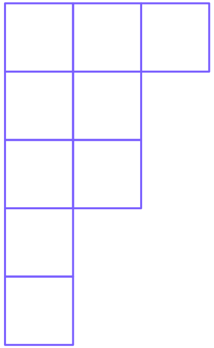




Making a shadow diagram (for π = 41325):

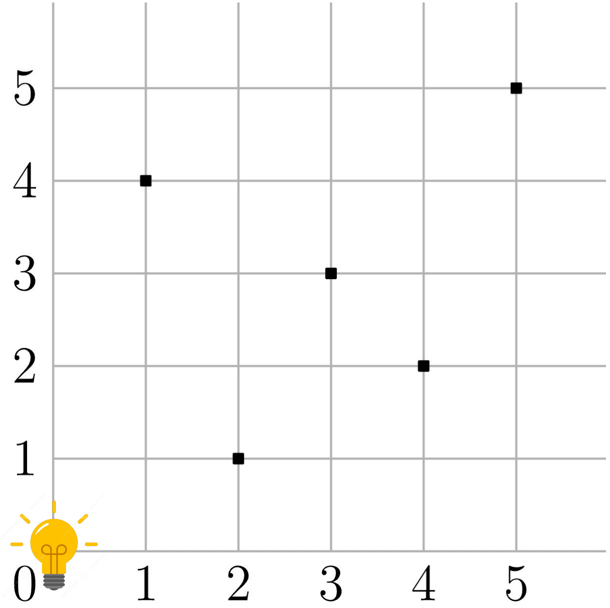
1) given $\pi = x_1 x_2 \ldots x_n$ represent x_i by a box with coordinates (i, x_i)

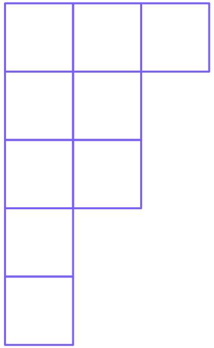




Making a shadow diagram (for $\pi = 41325$):

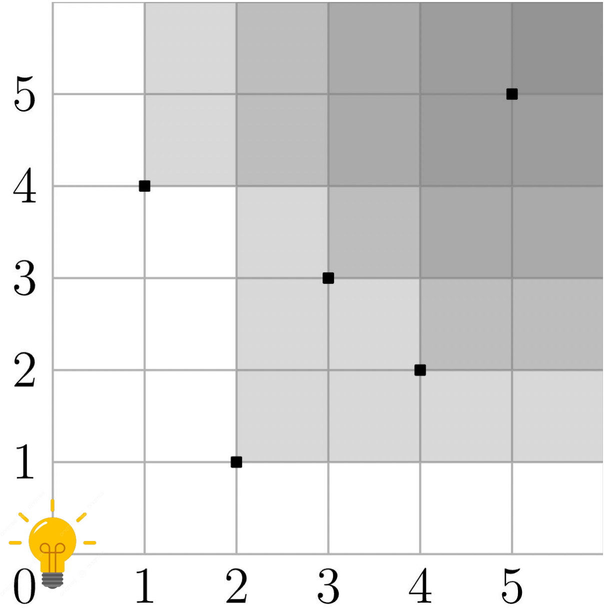
2) put a light at the origin

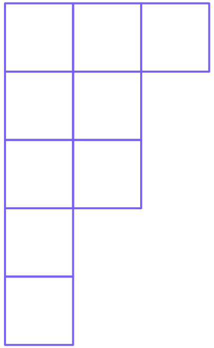




Making a shadow diagram (for $\pi = 41325$):

3) draw the shadow lines!





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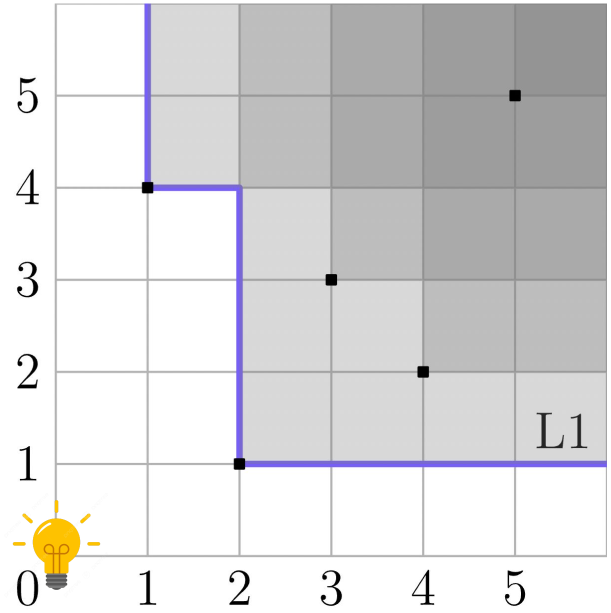

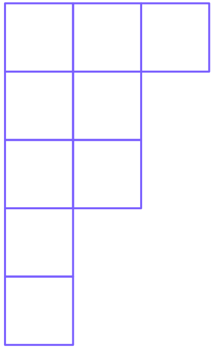

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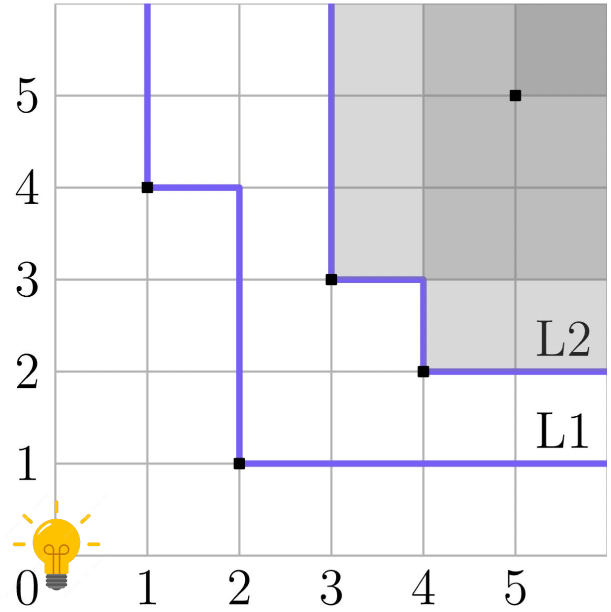
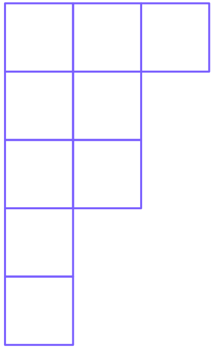
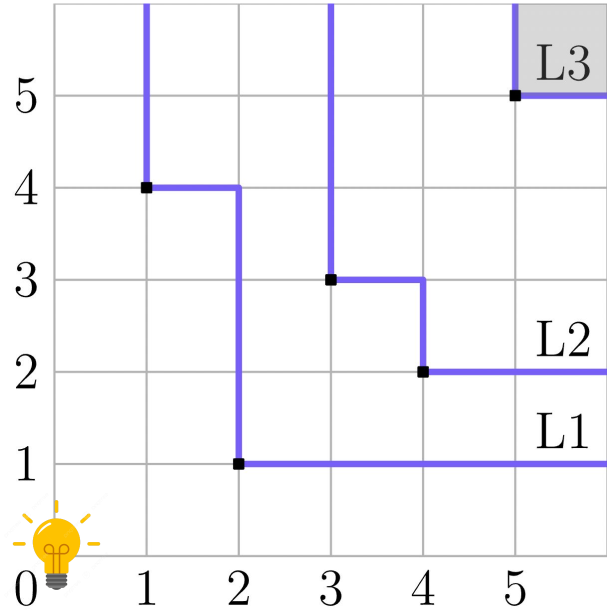
3) draw the shadow lines!

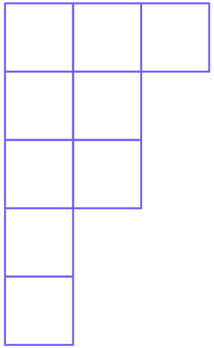




Making a shadow diagram (for $\pi = 41325$):

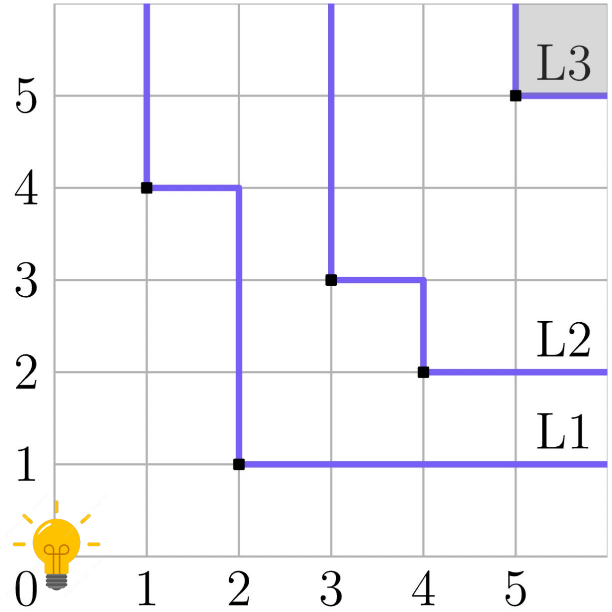
3) draw the shadow lines!

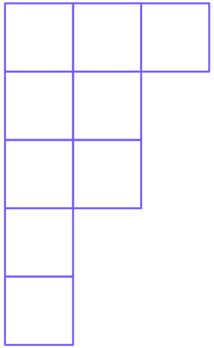




Making a shadow diagram (for $\pi = 41325$):

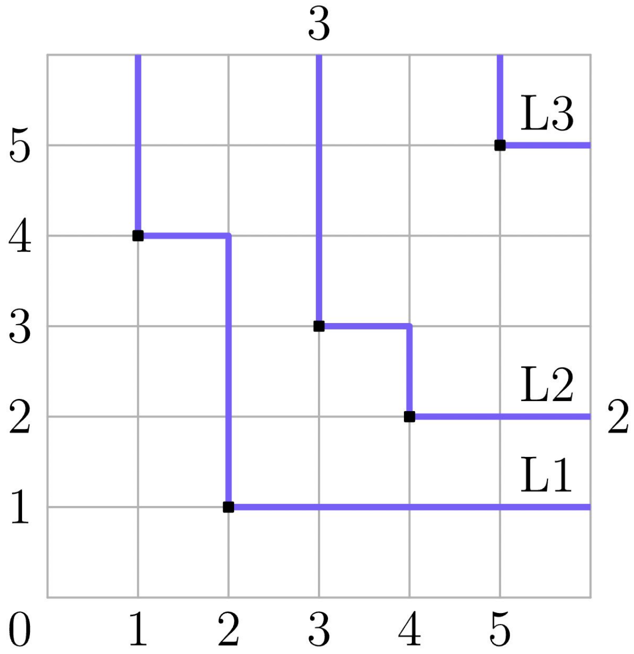
4) label shadow line coordinates – x_{L_i} and y_{L_i}

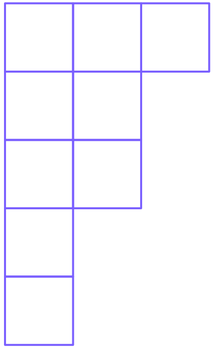




Making a shadow diagram (for π = 41325):

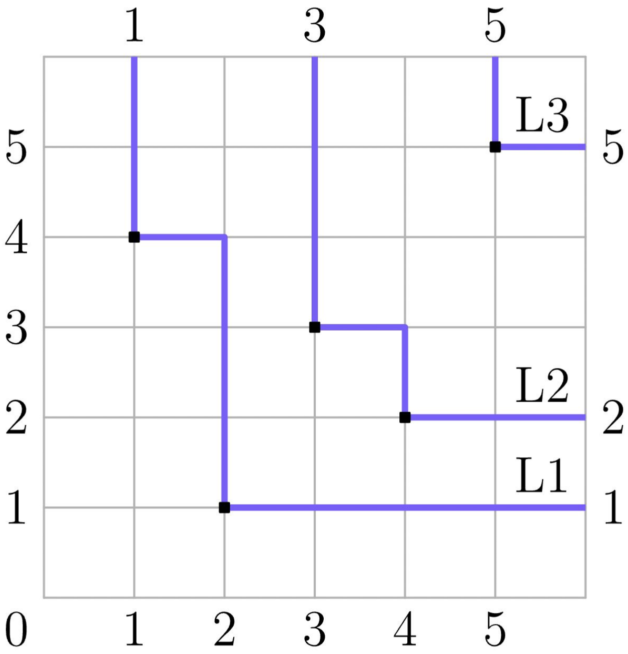
4) label shadow line coordinates – x_{L_i} and y_{L_i}

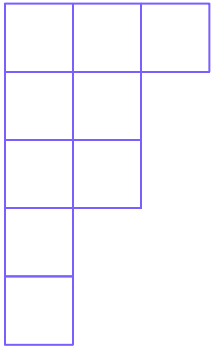




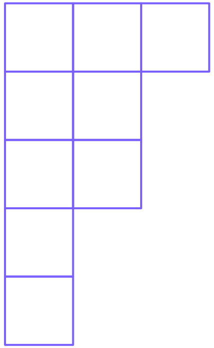
Making a shadow diagram (for π = 41325):

4) mark shadow line labels – x_{L_i} and y_{L_i}





Mathematics is the art of giving the same name to different things.



The secret correspondence ✨ (for $\pi = 41325$)

R-S algorithm:

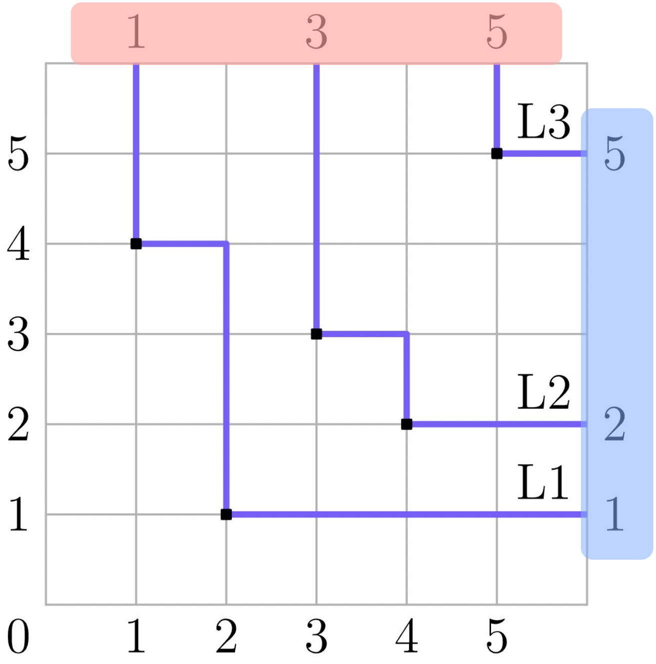
P=

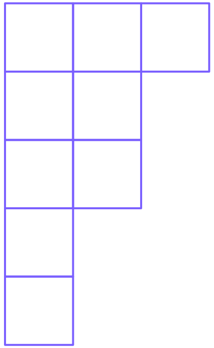
1	2	5
3		
4		

Q=

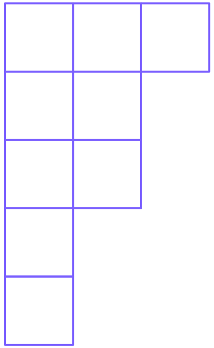
1	3	5
2		
4		

Viennot's construction:





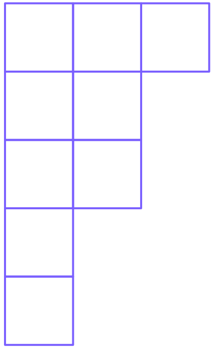
Definition (*i-th skeleton of π*):



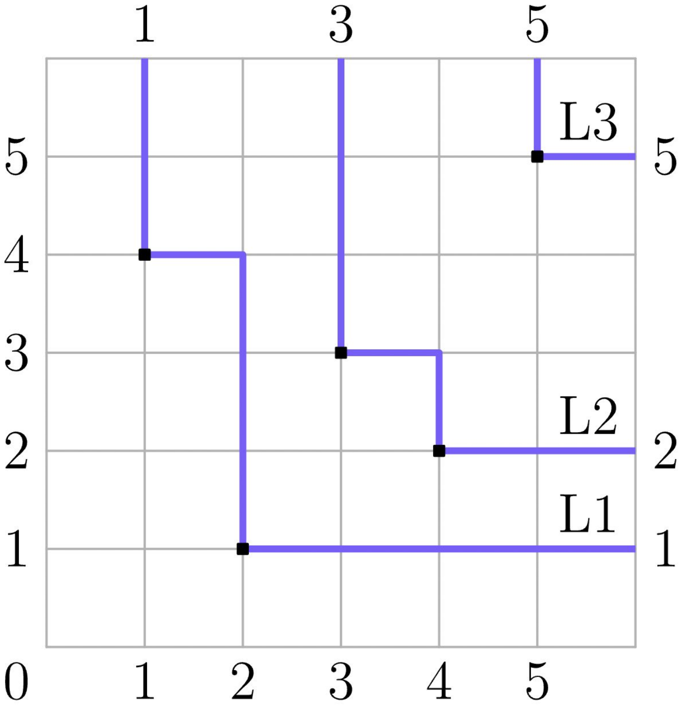
Definition (*i-th skeleton of π*): $\pi^{(1)} = \pi$ and

$$\pi^{(i)} = \begin{matrix} k_1 & k_2 & \dots & k_m \\ l_1 & l_2 & \dots & l_m \end{matrix},$$

where $(k_1, l_1), \dots, (k_m, l_m)$ are the north-east corners of the shadow diagram of $\pi^{(i-1)}$ listed in lexicographic order. The shadow lines of $\pi^{(i)}$ are denoted by $L_j^{(i)}$.



Definition (*i-th skeleton of π*):



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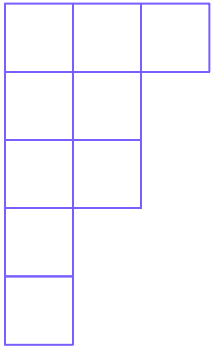
EXAMPLE

π = 41325



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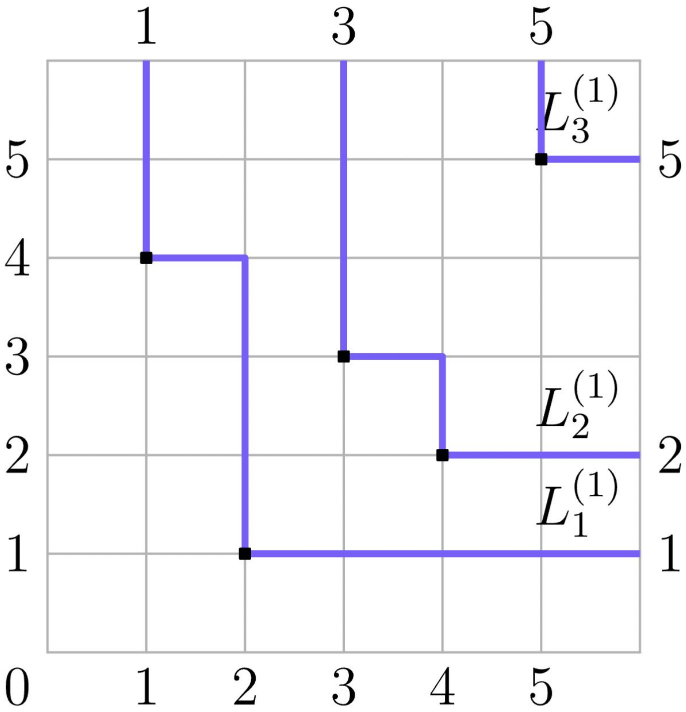
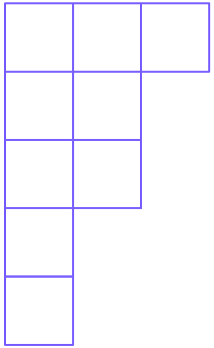


$$\pi^{(1)} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{matrix}$$

EXAMPLE

π = 41325



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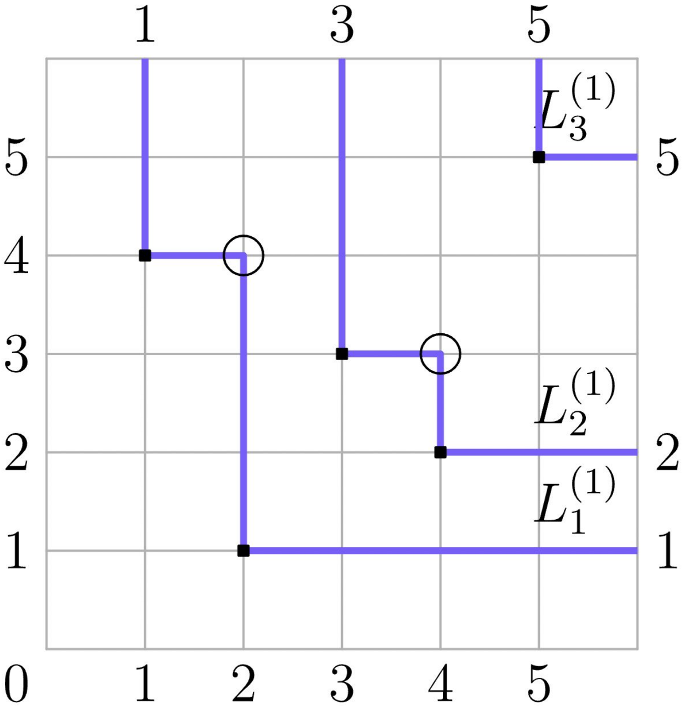
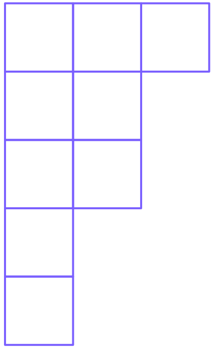


$$\pi^{(1)} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{matrix}$$

$$\pi^{(2)} = \begin{matrix} 2 & 4 \\ 4 & 3 \end{matrix}$$

EXAMPLE

π = 41325



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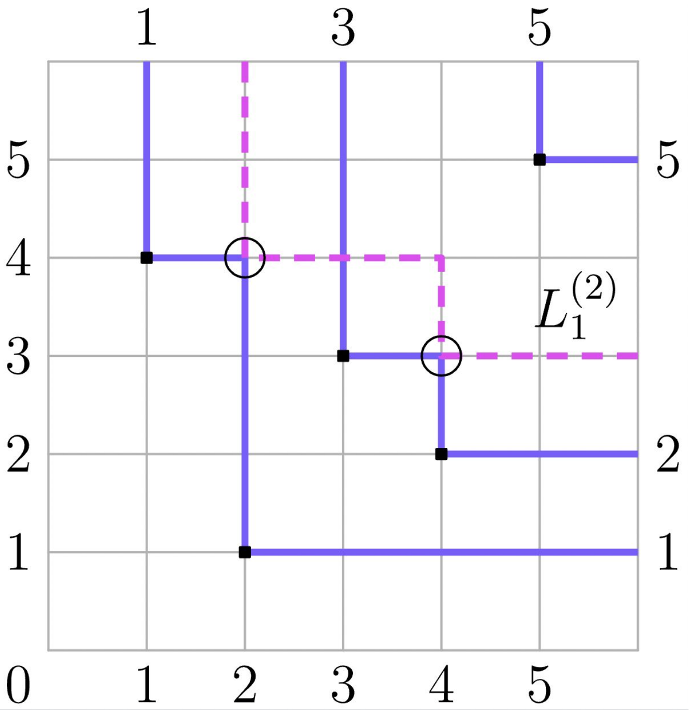

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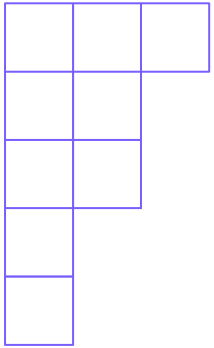
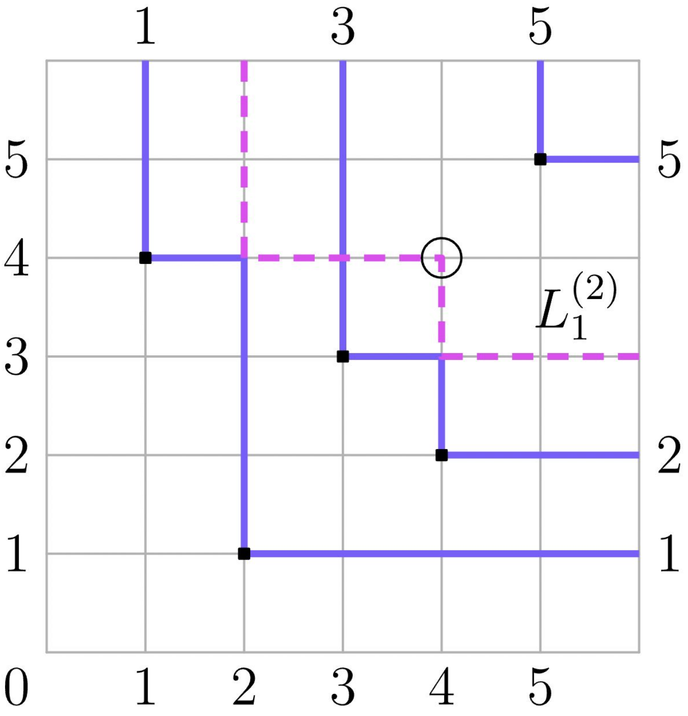


Definition (*i-th skeleton of π*):



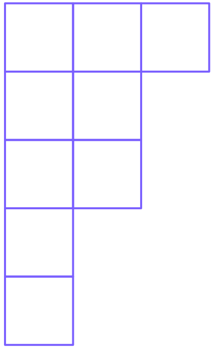
$$\pi^{(1)} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{matrix}$$

$$\pi^{(2)} = \begin{matrix} 2 & 4 \\ 4 & 3 \end{matrix}$$

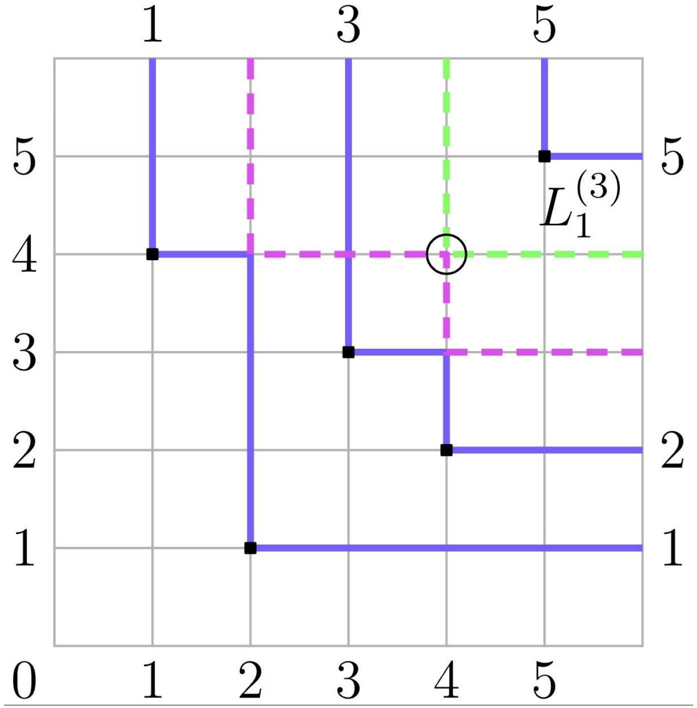
$$\pi^{(3)} = \begin{matrix} 4 \\ 4 \end{matrix}$$

EXAMPLE

π = 41325



Definition (*i-th skeleton of π*):



$$\pi^{(1)} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{matrix}$$

$$\pi^{(2)} = \begin{matrix} 2 & 4 \\ 4 & 3 \end{matrix}$$

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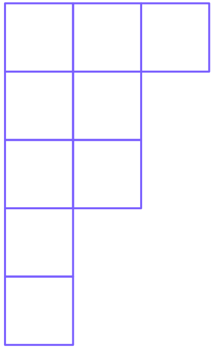
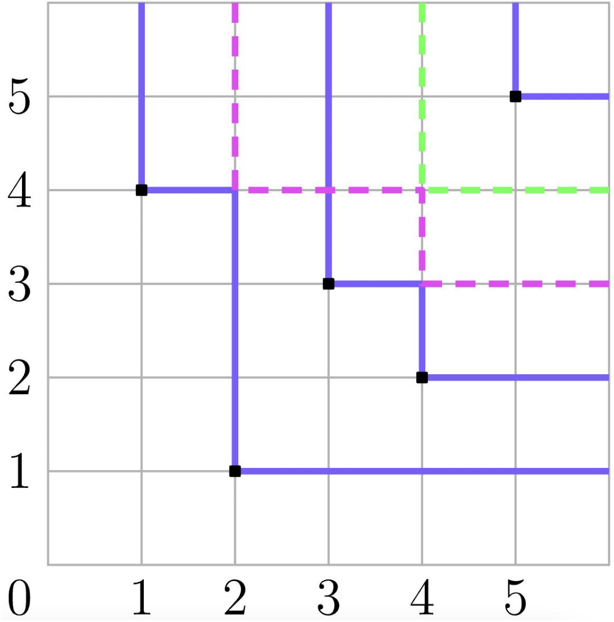
EXAMPLE

π = 41325



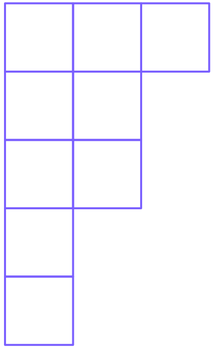
skeleton shadow diagram:



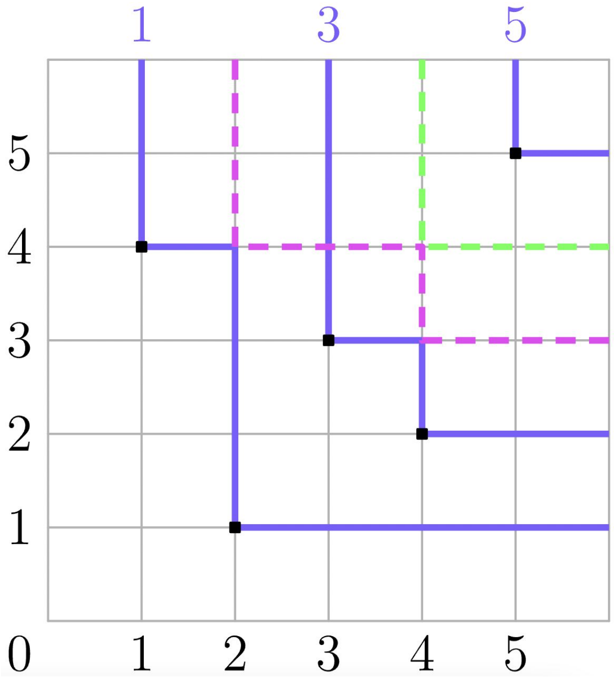
tableaux:

P=

Q=



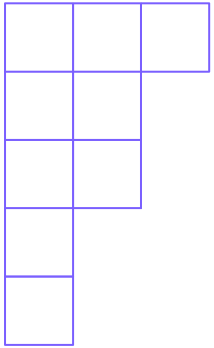
skeleton shadow diagram:



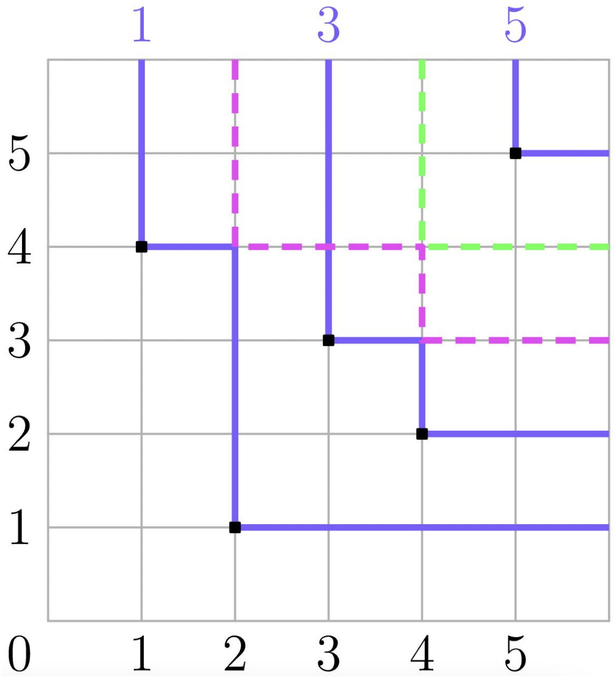
tableaux:

P=

Q=



skeleton shadow diagram:

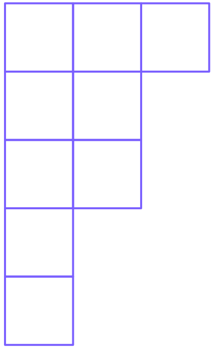


tableaux:

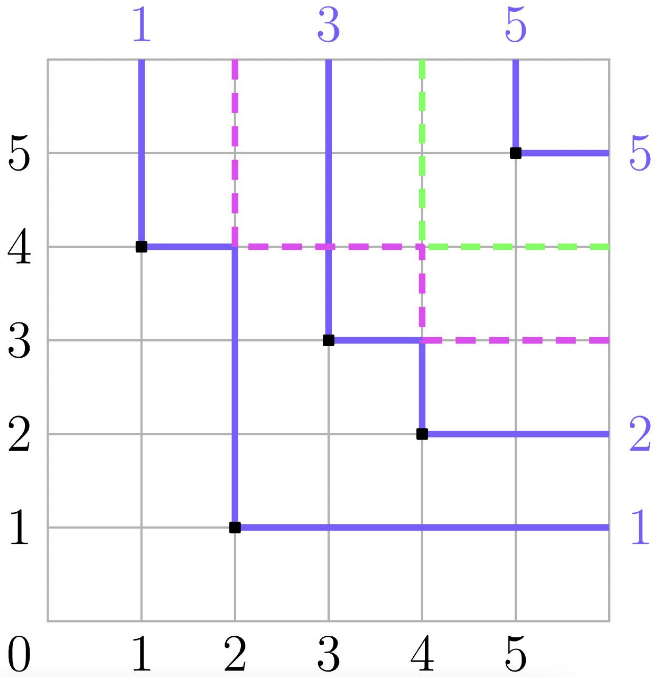
P=

Q=

1	3	5



skeleton shadow diagram:

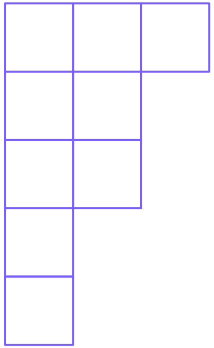


tableaux:

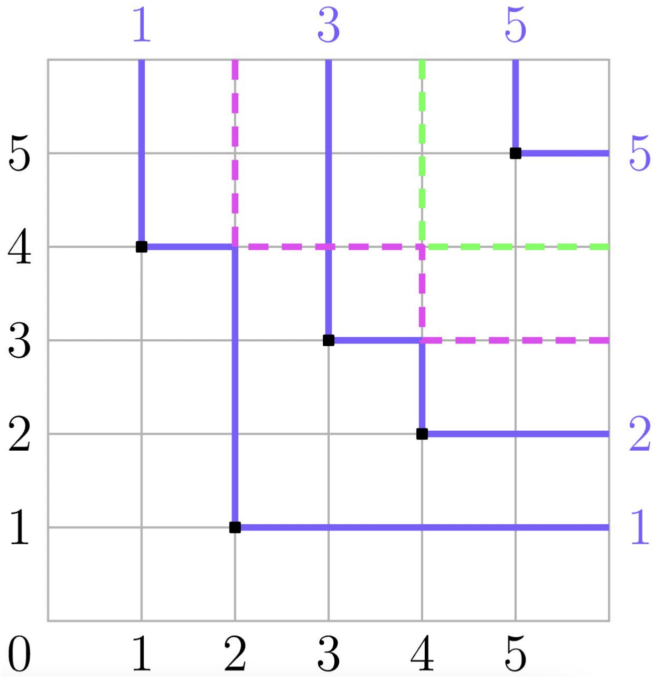
P=

Q=

1	3	5



skeleton shadow diagram:



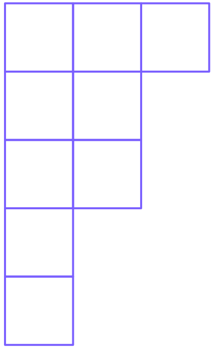
tableaux:

P =

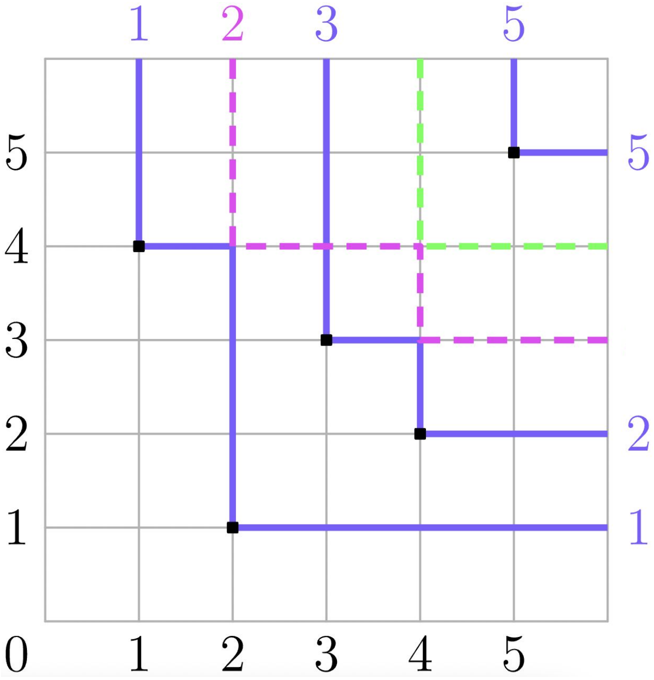
1	2	5

Q =

1	3	5



skeleton shadow diagram:



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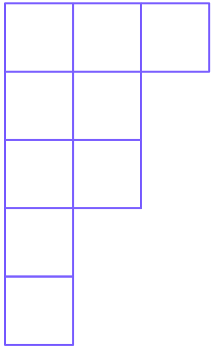
P =

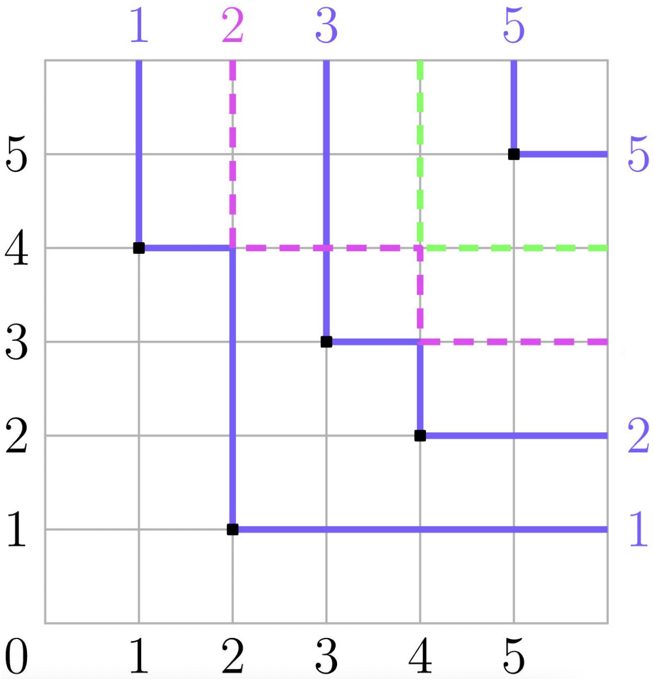
1	2	5

Q =

1	3	5



skeleton shadow diagram:



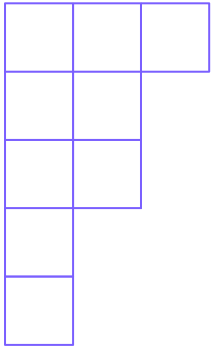
tableaux:

P=

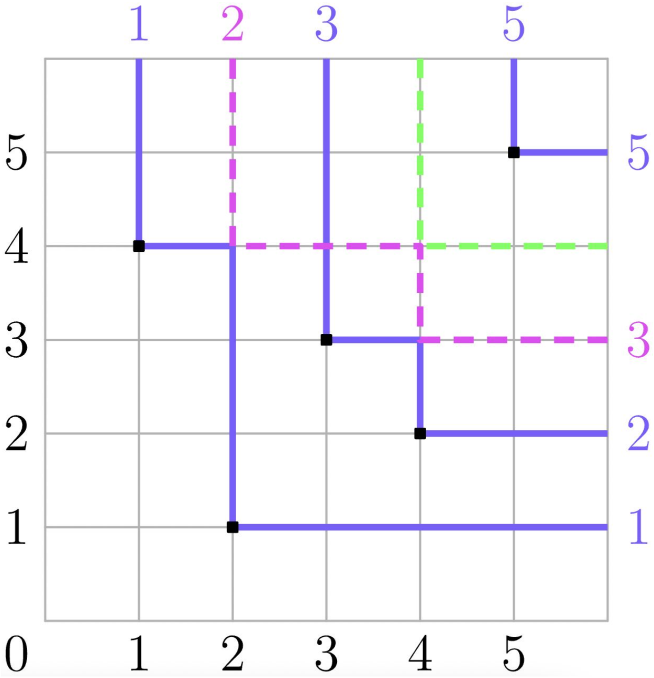
1	2	5

Q=

1	3	5
2		



skeleton shadow diagram:



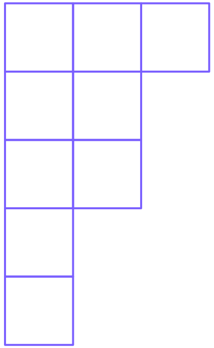
tableaux:

P =

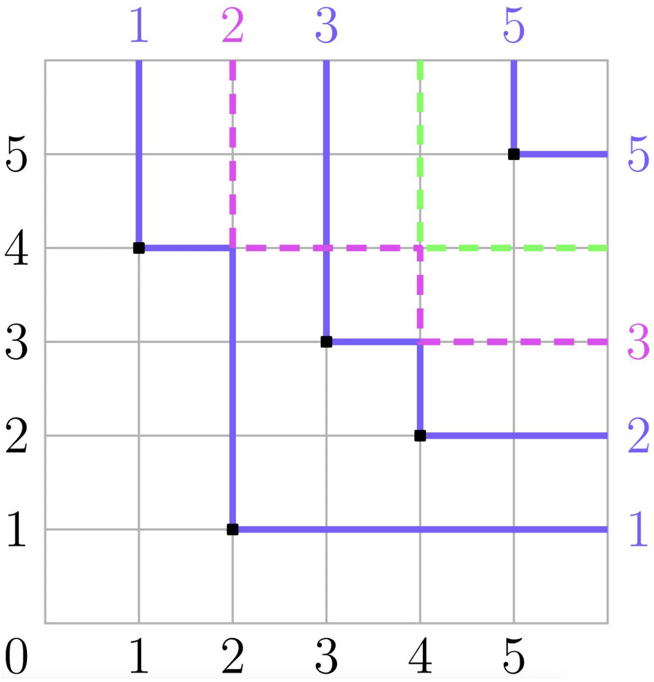
1	2	5

Q =

1	3	5
2		



skeleton shadow diagram:



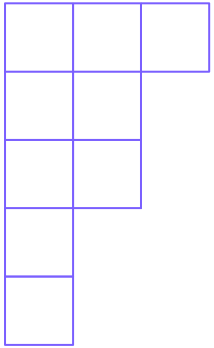
tableaux:

P=

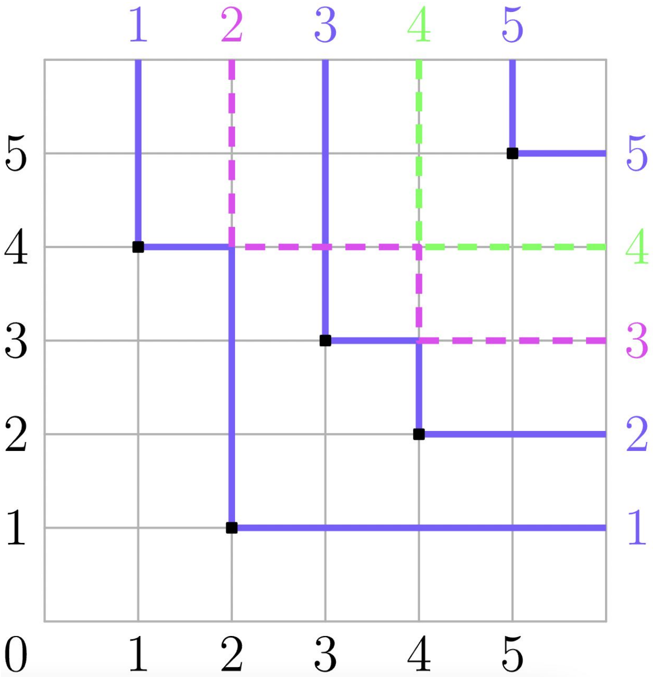
1	2	5
3		

Q=

1	3	5
2		



skeleton shadow diagram:



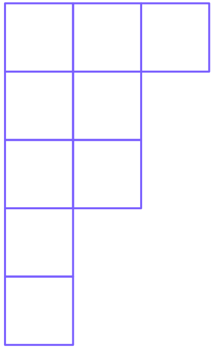
tableaux:

P=

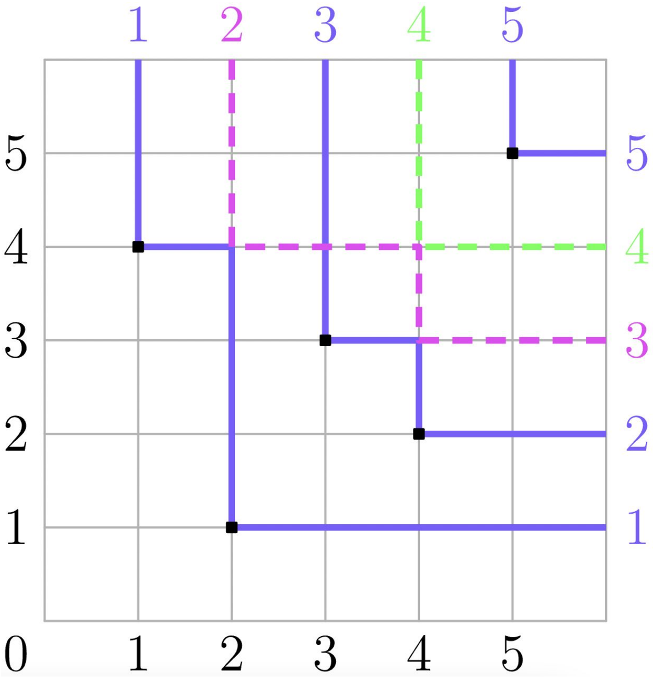
1	2	5
3		

Q=

1	3	5
2		



skeleton shadow diagram: → tableaux:



P =

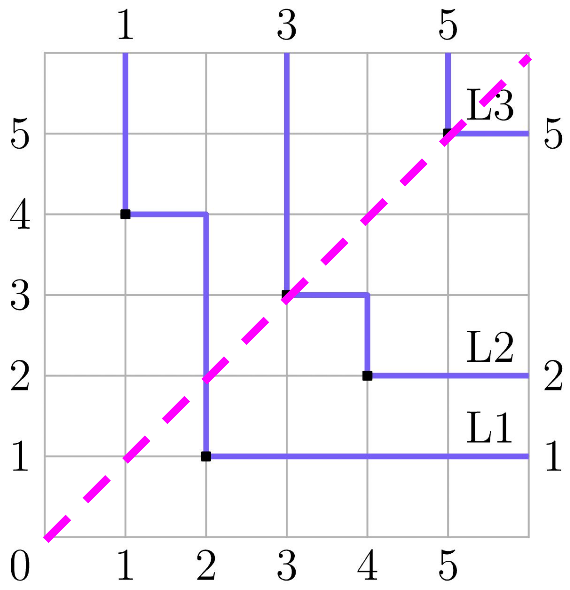
1	2	5
3		
4		

Q =

1	3	5
2		
4		

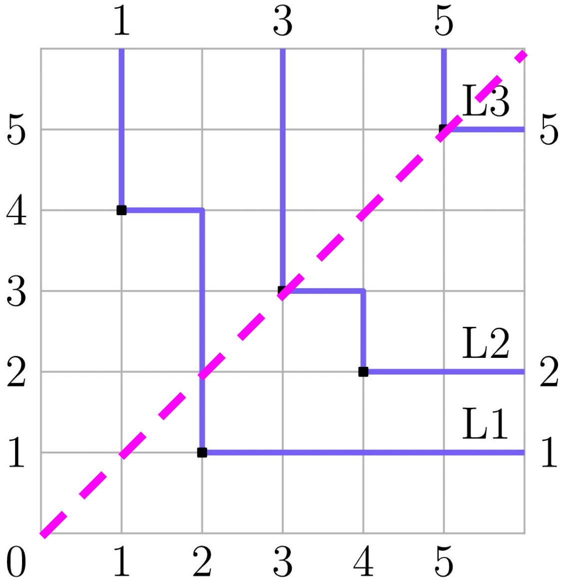
$\pi \rightarrow \pi^{-1}$

$(index, x_{index}) \rightarrow (x_{index}, index)$

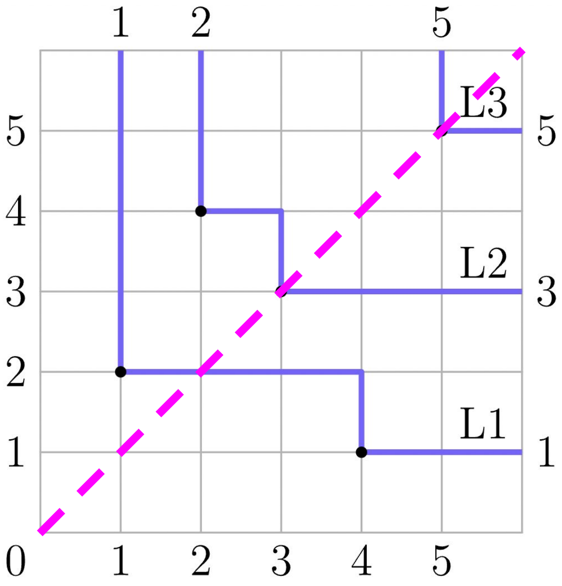


$\pi \rightarrow \pi^{-1}$

$(index, x_{index}) \rightarrow (x_{index}, index)$

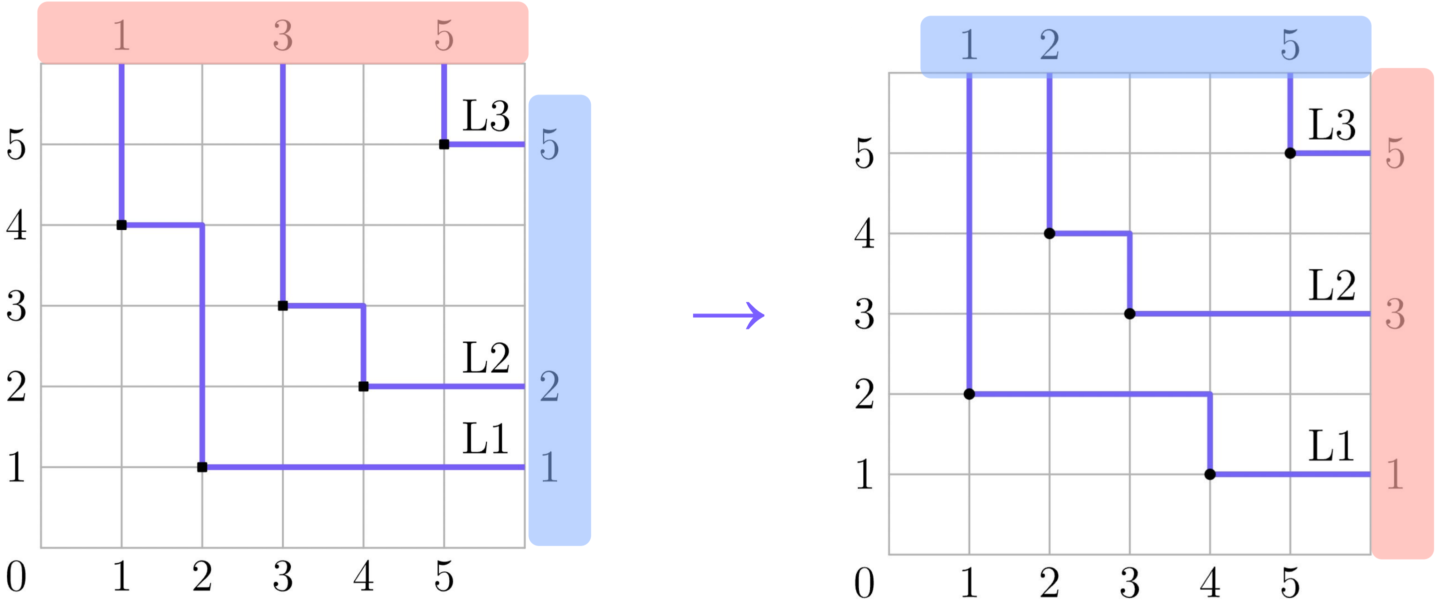


→



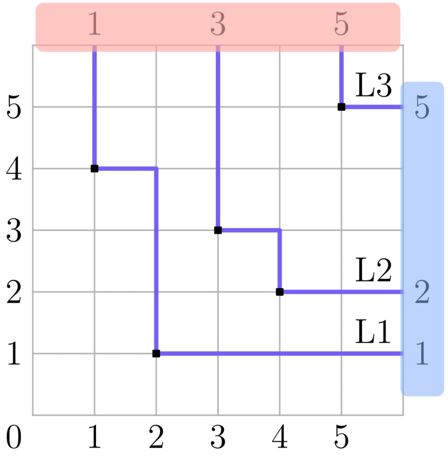
$\pi \rightarrow \pi^{-1}$

$(index, x_{index}) \rightarrow (x_{index}, index)$

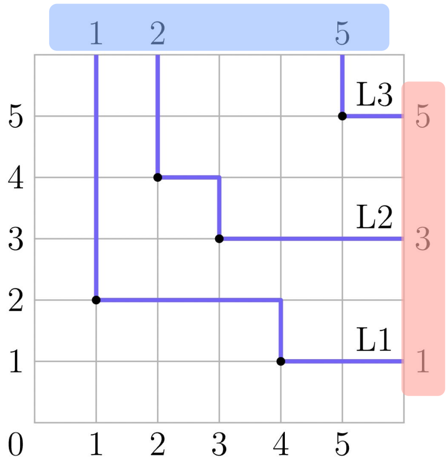


$\pi \rightarrow \pi^{-1}$

$(index, x_{index}) \rightarrow (x_{index}, index)$



$\rightarrow$



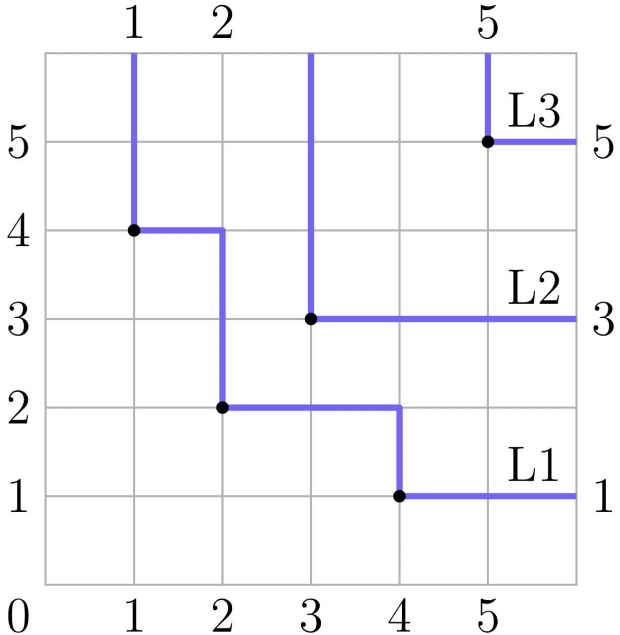
$P(\pi) \rightarrow Q(\pi^{-1})$

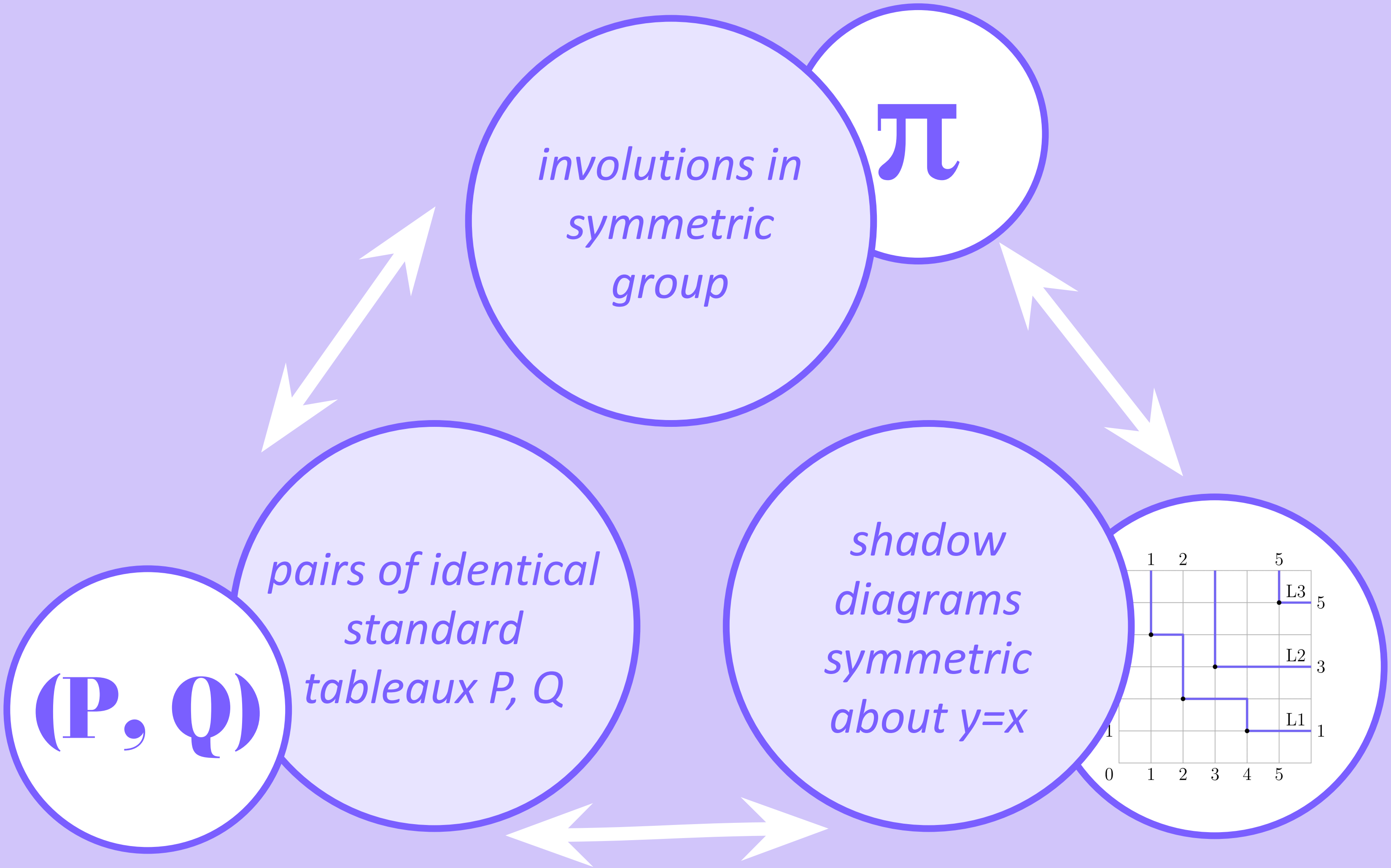
for *involution*, $\pi = \pi^{-1}$

$P(\pi) = Q(\pi^{-1}) \Rightarrow P(\pi) = Q(\pi)$ for π involution

then, there is a bijection from *involutions* to *pairs of identical standard tableaux* $(P(\pi), P(\pi))$

Note: involutions correspond to shadow diagrams that are *symmetric* about y=x




involutions in symmetric group
π
pairs of identical standard tableaux P, Q
(P, Q)
shadow diagrams symmetric about y=x
1
2
5
L3
5
L2
3
L1
1
1
0
1
2
3
4
5

 # involutions

equals

 # std tableau

Example with S_4

Standard Tableaux with 4 elements: 10

1	2	3	4

1	2
3	
4	

1	3
2	
4	

1	4
2	
3	

1	2
3	4

1	3
2	4

1
2
3
4

1	2	3
4		

1	3	4
2		

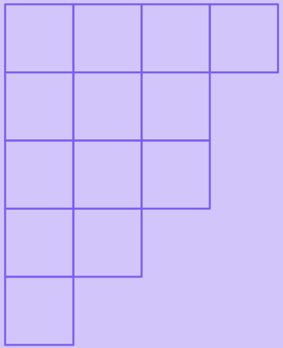
1	2	4
3		

Example with S_4

Involutions in S_4: 10

1234 1243 1432 1324 2134 3214 4231

2143 3412 4321



Thank you, PRIMES and parents!

We would like to thank our mentor, Serina Hu, PRIMES coordinators Prof. Pavel Etingof, Dr. Slava Gerovitch, and Dr. Tanya Khovanova, everybody behind the PRIMES program, as well as our parents!